MANONMANIAM SUNDARANAR UNIVERSITY

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TAMILNADU STATE

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

GRAPH THEORY AND COMBINATORICS

M.Sc. MATHEMATICS

FIRST YEAR

GRAPH THEORY AND COMBINATORICS

Syllabus

Unit I: Graphs and subgraphs- Trees- Cut edges - Cut vertices - Cayley's formula.

Unit II: Connectivity - Blocks - Euler tours and Hamilton cycles: Euler tours - Hamilton cycles.

Unit III: Matchings - Edge colorings.

Unit IV: Independent sets and cliques -Vertex colorings .

Unit V : Permutations and Combinations -Generating functions -The principle of inclusion and exclusion.

Text Books:

1. J.A.Bondy and U.S.R.Murty, Graph Theory with Applications, North Holland, Amsterdam, 1982.

Unit I: Chapter 1 (except 1.8,1.9), Chapter 2 (except 2.5)

Unit II: Chapter 3 (except 3.3.), Chapter 4 (except 4.3 and 4.4)

Unit III: Chapter 5 (except 5.4 and 5.5), Chapter 6 (except 6.3)

Unit IV: Chapter 7 (except 7.4 and 7.5) Chapter 8 (except 8.6)

2. C.L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, Inc. 1968.

Unit V : Chapter 1 (except 1.7, 1.8), Chapter 2 (except 2.6, 2.7, 2.8), Chapter 4 (only sections 4.1 and 4.2)

Contents

Chapter 1

GRAPHS AND SUBGRAPHS

In recent years, graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and chemistry to genetics and linguistics, and from electrical engineering and geography to sociology and architecture. At the same time it has also emerged as a worthwhile mathematical discipline in its own right.

The origin of graph theory can be traced to Leonhard Euler who devised in 1735 a problem that came to be known as the "Seven Bridges of Konigsberg". In this problem, someone had to cross once all the bridges only once and in a continuous sequence, a problem the Euler proved to have no solution by representing it as a set of nodes and links. This led the the foundation of graph theory and its subsequent improvements.

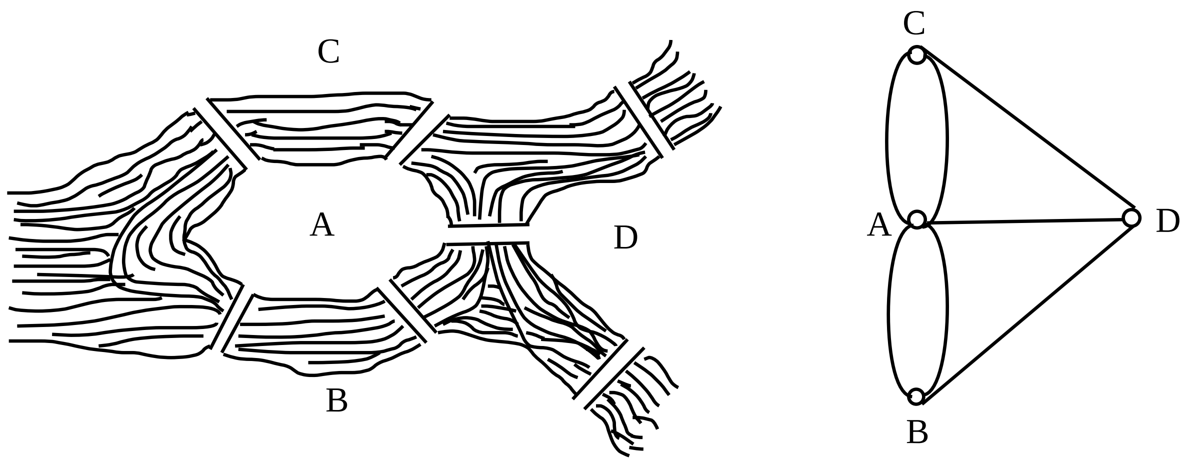


Figure 1.1. Geographic Map: The Konigsberg Bridges

Many real-world situations can conveniently be described by means of diagram consisting of a set of points together with lines joining certain pairs of these points. For example, the points could represent people, with lines joining pairs of friends; or the points might be communication centres, with lines representing communication links. In sucg diagrams one is mainly interested in whether or not two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of graph.

1.1 Graphs and simple graphs

Definition 1.1.1. A *graph* G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of

(i) a nonempty set $V(G)$ of vertices

(ii) a set $E(G)$ disjoint from $V(G)$, of edges

and (iii) an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to *join* u and v. The vertices u and v are called the *ends* of e.

Example 1.1.2. Let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and ψ_G be defined by $\psi_G(e_1) = v_1v_2$, $\psi_G(e_2) = v_2v_2$,
$\psi_G(e_3) = v_2v_5$, $\psi_G(e_4) = v_1v_3$,
$\psi_G(e_5) = v_1v_5$, $\psi_G(e_6) = v_5v_3$,
$\psi_G(e_7) = v_3v_4$ Then $(V(G), E(G), \psi_G)$ is a graph.

Example 1.1.3. Let $V(H) = \{u, v, w, x, y\}$, $E(H) = \{a, b, c, d, e, \}$ and ψ_H be defined by $\psi_H(a) = uv$, $\psi_H(b) = vw$, $\psi_H(c) = wx$, $\psi_H(d) = xy$, $\psi_H(e) = vy$, $\psi_H(f) = vx$ Then $(V(H), E(H), \psi_H)$ is a graph.

Diagrammatic representation of a graph

Graphs are so named because they can be represented graphically and thus many of its properties can be understood. Each vertex is indicated by a point and each edge by a line joining the points which represent its ends. Figures $1.1.1$ and $1.1.2$ are diagrammatic representations of the graphs in Examples 1.1.2 and 1.1.3 respectively.

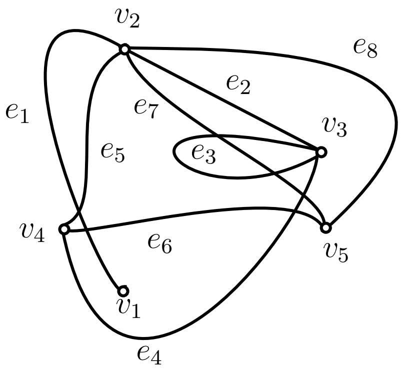


Figure 1.1.1

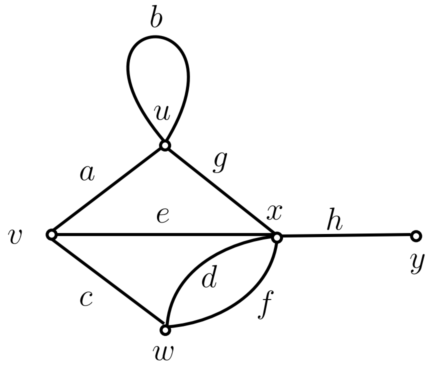


Figure 1.1.2

Remark 1.1.4. 1. There is no unique way of drawing a graph.

2. The relative position of points representing vertices and lines representing the edges have no significance.

3. A digram of a graph merely depicts the incidence relation holding between its vertices and edges.

The graph shown in Figure 1.1.2 can also be represented as in Figure 1.1.3.

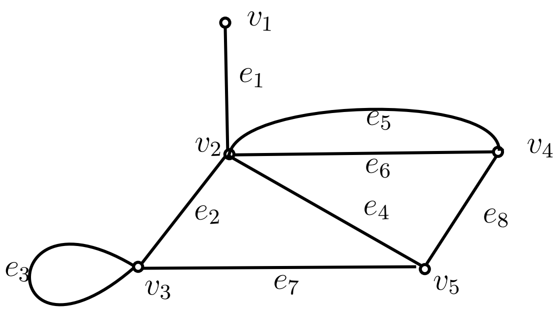


Figure 1.1.3

Definition 1.1.5. If $e = uv$, is an edge, then u and v are said to be *incident* with the edge e and vice versa. Also u and v are called *adjacent vertices*. Two edges which are incident with a common neighbour are called *adjacent edges*.

For example, v_3 and v_4 are adjacent vertices, since they are incident with the common edge e_4. And e_2 and e_1 are adjacent edges, since they are incident with the common vertex v_2.

An edge with identical ends is called a *loop*. For example, the edge e_3 in Figure $1.1.3$ is a loop.

An edge with distinct ends are called links.

Links that have the same pair of vertices are called *multiple edges* or *parallel edges*. For example, the edges e_7 and e_8 are parallel edges.

A graph is *simple* if it has no loops or multiple edges. The graph G given in Figure $1.1.3$ is not a simple graph. A simple graph is given in Figure $1.1.4$.

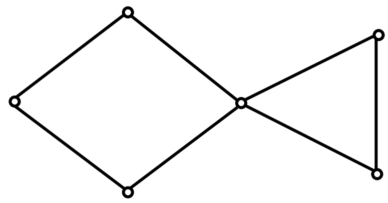


Figure $1.1.4$ Simple Graph

Definition 1.1.6. A graph is *finite* if both its vertex set and edge set are finite. *All graphs considered in this book are finite.*

A graph with just one vertex is called a *trivial graph* and all other graphs are called as *nontrivial graphs*.

Notation. The number of vertices in a graph is denoted by $\nu(G)$ or simply ν. The number of edges in a graph is denoted by $\epsilon(G)$ or simply ϵ. The vertex set and the edge set of G are simply denoted by V and E, respectively.

1.2 Isomorphism

Definition 1.2.1. Two graphs G and H are *identical* if

$(i)\ V(G) = V(H)$

$(ii)\ E(G) = E(H)$ and

$(iii)\ \psi_G = \psi_H$

Then we write $G = H$.

If two graphs are identical, then they can be clearly represented by identical diagrams. However, it is possible for graphs that not identical to have essentially the same diagram. For example, the diagrams in Figures $1.1.2$ and $1.1.3$ are same in structure but not in labels. Hence they are not identical but are isomorphic.

Definition 1.2.2. Let G and H be two graphs. Let $\theta : V(G) \to V(H)$ and $N : E(G) \to E(H)$ be two bijections such that $\psi_G(e) = uv$ if and only if $\psi_H(N(e)) = \theta(u)\theta(v)$. Then the pair (θ, N) is an *isomorphism* between G and H.

Graphs G and H are *isomorphic* if there is an isomorphism between G and H; in this case, we write $G \cong H$.

Remark 1.2.3. Clearly G and H has the same structure and differs only in the names of vertices and edges. Since we are interested in the structural properties of graphs, we shall often omit labels while drawing graphs. An unlabeled graph can be thought of as a representative of an equivalence class of isomorphic graphs. We assign labels to the vertices and edges in a graph mainly for the purpose of referring to them.

The two graphs shown in Figure 1.2.1 are isomorphic.

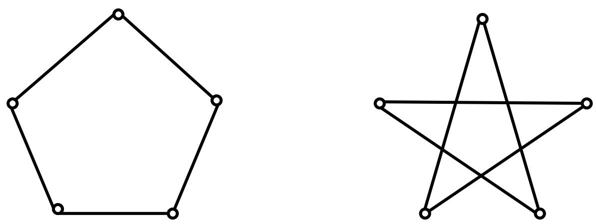
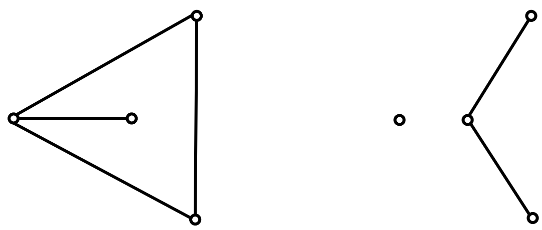


Figure 1.2.1 Isomorphic Graphs

Definition 1.2.4. Let G be a simple graph. The *complement* of G is the simple graph with the same vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. It is denoted by G^c.

For example, a graph G and its complement G^c are given in Figure 1.2.2.



G G^c

Figure 1.2.2

Definition 1.2.5. A graph G is said to be *self complementary* if $G \cong G^c$.

For example, in Figure 1.2.3, the graph G is a self complementary graph on 4 vertices and the graph H is a self complementary graph on 5 vertices.

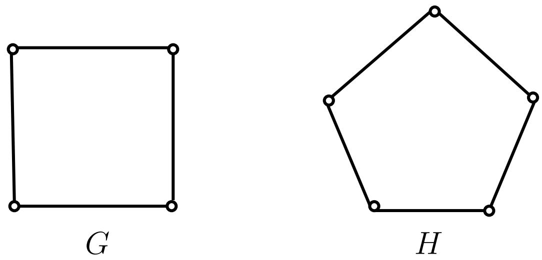


Figure 1.2.3

Remark 1.2.6. If G is self complementary, then
$|V(G)| = |V(G^c)|$ and $|E(G)| = |E(G^c)|$.

Some special classes of graphs

Definition 1.2.7. A simple graph in which every pair of distinct vertices is joined by an edge is called a *complete graph*. There is just one complete graph on n vertices up to isomorphism and is denoted by K_n.

For example, the complete graphs on 1, 2, 3, 4 and 5 vertices are given in Figure 1.2.4.

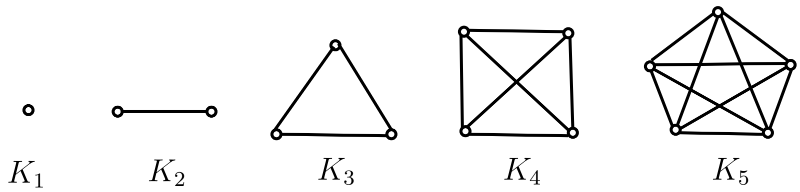
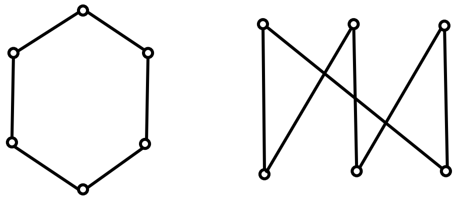


Figure 1.2.4

Definition 1.2.8. A graph whose vertex set can be partitioned into two subsets V_1 and V_2 such that each edge has one end in V_1 and another end in V_2, is called a *bipartite graph*. Such a partition (V_1, V_2) is called a *bipartition* of G.

The two graphs given in Figure $1.2.5$ are bipartite graphs.

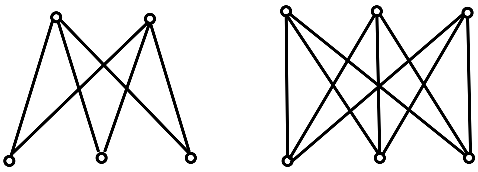


Bipartite Graphs

Figure 1.2.5

Definition 1.2.9. A simple bipartite graph with partition (V_1, V_2) in which every vertex of V_1 is joined to every vertex of V_2 is called a *complete bipartite graph.* If $|V_1| = m$ and $|V_2 = n|,$ then such a graph is denoted by $K_{m,n}.$

For example, $K_{2,3}$ and $K_{3,3}$ are given in Figure 1.2.6.

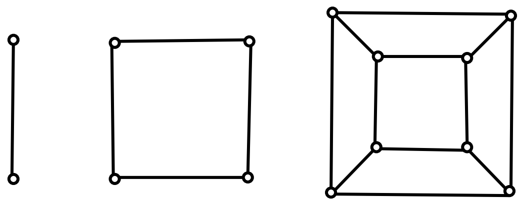


$K_{2,3}$ $K_{3,3}$

Figure 1.2.6

Definition 1.2.10. A graph whose vertices are the $k-$ tuples of $0'$ s and $1'$ s in which two vertices are joined if and only if they differ in exactly one coordinate is called a $k-$ **cube** and is denoted by $Q_k.$

For example, $1-$ cube, $2-$ cube and $3-$ cube are given in Figure 1.2.7 .



Q_1 Q_2 Q_3

Figure 1.2.7

Solved Problems

Problem 1. Prove that the number of simple even graphs (degree of all vertices is even) with n vertices is $2^{\binom{n-1}{2}}$.

Solution. There is a bijection between simple graphs with $n-1$ vertices and even simple graphs on n vertices. Given a simple graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$ we can construct a even simple graph of n vertices. We know that no of vertices of odd degree is even. Construct a new graph G^* with $V(G^*) = V(G) \cup \{v_n\}$ and $E(G^*) = E(G) \cup \{v_i v_n : v_i \in V(G),\ deg_G(v_i) \text{ is odd}\}$. Then G^* is a even simple graph.

Conversely, given an even simple graph G^* we will get back G by $G^* - v_n$. Since in a simple graph of $n-1$ vertices can have atmost $\binom{n-1}{2}$ edges thus no of even simple graph of n vertices is $2^{\binom{n-1}{2}}$. ■

Problem 2. If G is simple, prove that $\epsilon \leq \binom{\nu}{2}$. Also prove that $\epsilon = \binom{\nu}{2}$ if and only if G is complete.

Solution. Since G is simple, every edge of G is incident with two vertices. Hence the number of edges cannot exceed the number of ways of selecting two distinct vertices from ν vertices. Thus,

$$\begin{aligned} \epsilon &\leq \text{ number of ways of choosing two vertices from } \nu \text{ vertices} \\ &= \binom{\nu}{2} \end{aligned}$$

Also, $\epsilon = \binom{\nu}{2} \Leftrightarrow$ the edge set contains all pairs of distinct vertices

$\Leftrightarrow$ any two distinct vertices are adjacent in G

$\Leftrightarrow$ G is complete. ■

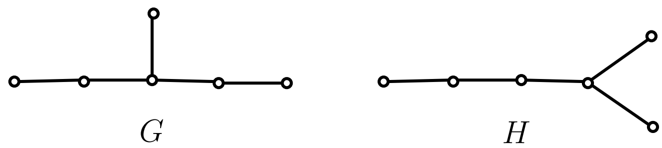
Problem 3. If $G \cong H$, prove that $\nu(G) = \nu(H)$ and $\epsilon(G) = \epsilon(H)$. Give an example to show that the converse is not true.

Solution. Since G is isomorphic to H, there exist bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$.

Hence $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$.

Therefore, $\nu(G) = \nu(H)$ and $\epsilon(G) = \epsilon(H)$. To prove the converse part is not true, consider the graphs G and H shown in Figure 1.2.8.

Clearly $\nu(G) = \nu(H)$ and $\epsilon(G) = \epsilon(H)$; but G and H are not isomorphic because the neighbours of the end vertices of G are mutually distinct whereas the neighbours of two end vertices of H are same. ■



Two nonsiomorphic graphs of same order and size

Figure 1.2.8

Problem 4. Show that $\epsilon(K_{m,n}) = mn$.

Solution Since $K_{m,n}$ is a bipartite graph, it has a bipartition (V_1, V_2) with $|V_1| = m$ and $|V_2| = n$. Since G is a complete bipartite graph, it contains all edges with one end in V_1 and the other end in V_2. Hence the number of edges of $K_{m,n}$ is equal to the sum of the number of edges incident with the vertices of V_1.

$$\begin{aligned}\epsilon(K_{m,n}) &= n + n + \ldots + n, (m \text{ times}) \text{ (since } |V_1| = m \text{)}\\ &= mn\end{aligned}$$ ■

Problem 5. If G is simple and bipartite, prove that $\epsilon \leq \dfrac{\nu^2}{4}$.

Solution Let (V_1, V_2) be a bipartition of G with $|V_1| = m$ and $|V_2| = n$ and $\nu(G) = m + n$.

Each vertex in V_1 is adjacent to at most $|V_2| = n$ vertices. Thus,

$$\begin{aligned}\epsilon &\leq \text{ number of edges incident with vertices of } V_1.\\ &\leq n + n + \ldots + n \ (m \text{ times})\\ &= mn\\ &\leq \left(\frac{m+n}{2}\right)^2\end{aligned}$$

$$= \frac{(m+n)^2}{4}$$
$$= \frac{\nu^2}{4}. \qquad \blacksquare$$

Problem 6. Prove that the k -cube has 2^k vertices, $k2^{k-1}$ edges and is bipartite.

Solution. Clearly, $V(Q_k)$ is the set of all ordered k -tuples of $0'$ s and $1'$ s. Number of such tuples is 2^k

Therefore, $\nu(Q_k) = 2^k$

Since two vertices are joined if and only if they differ in exactly one coordinate, it follows that each vertex is adjacent to exactly k vertices. Thus,

$$\epsilon(Q_k) = \frac{k + k + \ldots + k \ (2^k \text{ times})}{2}, \text{ since each edge is incident with two vertices.}$$
$$= k.\frac{2^k}{2}$$
$$= k2^{k-1}$$

Now, let $X = \{$k-tuples with even number of 0's$\}$

$Y = \{$k-tuples with odd number of 0's$\}$. Now,

$X \cup Y = Q_k$ and $X \cap Y = \phi$

Also, any two vertices of $X(\text{or } Y)$ differ at least in two coordinates and hence they are not adjacent. Thus any edge must have one end in X and the other end in Y. Thus (X, Y) is a bipartition of Q_k, which completes the proof. $\blacksquare$

Problem 7. If G is self complementary, prove that $\nu = 0 \ or \ 1(mod \ 4)$.

Solution Since G is self complementary, $G \cong G^c$. Therefore,

$|E(G)| = |E(G^c)|$ and

$E(G) \cup E(G^c) = E(K_\nu)$

Thus, $|E(G)| + |E(G^c)| = \binom{\nu}{2}$

$$\Rightarrow 2|E(G)| = \frac{\nu(\nu-1)}{2}$$
$$\Rightarrow \frac{\nu(\nu-1)}{4} = |E(G)|, \text{ which is an integer}$$

$\Rightarrow \nu \; or \; \nu - 1$ is a multiple of $4.$

Thus, $\nu \equiv 0, 1 \; (mod \; 4).$ ■

Exercises

1. List five situations from everyday life in which graphs arise naturally.
2. Draw all simple graphs on 3 vertices.
3. Prove that there are eleven nonisomorphic simple graphs on 4 vertices.
4. Prove that two simple graphs G and H are isomorphic if and only if there is a bijection $\theta : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\theta(u)\theta(v) \in E(H).$

1.3 Subgraphs

Definition 1.3.1. A graph $H = (V(H), E(H), \psi_H)$ is called a *subgraph* of the graph $G = (V(G), E(G), \psi_G)$ if

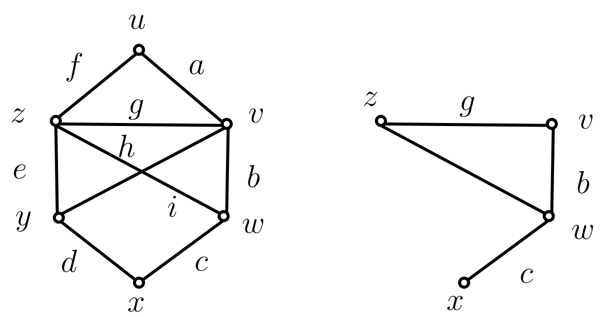
$(i) \; V(H) \subseteq V(G);$

$(ii) \; E(H) \subseteq E(G);$ and

$(iii) \; \psi_H$ is a restriction of ψ_H to $E(H).$

In this case, we write $H \subseteq G.$ If $H \subseteq G$ but $H \neq G,$ then H is called a *proper subgraph* of G and we write $H \subset G.$ If H is a subgraph of $G,$ then G is called a *super graph* of $H.$ A *spanning subgraph* (or *spanning super graph*) of G is a subgraph (or super graph) H with $V(H) = V(G).$

A graph G and its proper subgraph H are given in Figure $1.3.1.$ Since $V(G) = V(H),$ H is also a spanning subgraph of $G.$

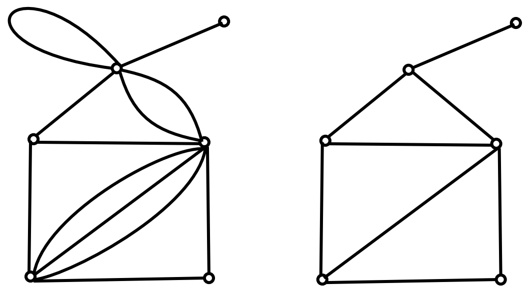


A graph G A spanning subgraph of G

Figure $1.3.1$

Definition 1.3.2. A graph obtained from a graph G by deleting all its loops and all its multiple edges except one, is called the underlying simple graph of $G.$

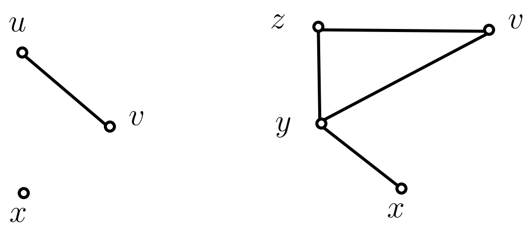
The underlying graph of the above graph is shown in Figure $1.3.2.$



A graph G Underlying graph of G

Figure $1.3.2$

Definition 1.3.3. Let G be a graph and V' be a nonempty subset of $V.$ The subgraph of G whose vertex set is V' and whose edge set is the set of those edges that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V'].$ We say that $G[V']$ is the induced subgraph of $G.$ The induced subgraph $G[V\backslash V']$ is denoted by $G - V'.$ It is the subgraph obtained from G by deleting the vertices in V' together with their incident edges. If $V' = \{v\},$ we write $G - v$ for $G - \{v\}.$



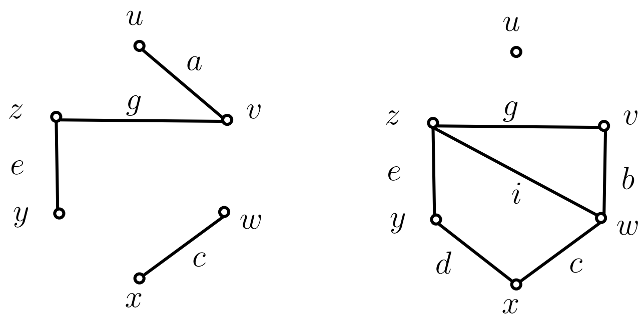
$G[\{u, v, x\}]$ $G - \{u, w\}$

Figure $1.3.3$

Figure $1.3.3$ shows the induced subgraph $G[\{u, v, x\}]$ and the vertex deleted subgraph $G - \{u, w\}$ of the graph G in Figure $1.3.1.$

Definition 1.3.4. Let G be a graph and E' be a nonempty subset of $E.$ The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G induced by E' and is denoted by $G[E']$ and is called the edge induced subgraph of $G.$ The spanning subgraph of G with edge set $E - E'$ is simply written as $G - E'.$ It is the subgraph obtained from G by deleting the edges in $E'.$

Figure 1.3.4 shows the edge-induced subgraph $G[\{a,c,e,g\}]$ and the edge deleted subgraph $G-\{a,b,f,i\}$ of the graph G Figure 1.3.1.



$G[\{a,c,e,g\}]$ $G-\{a,b,f,i\}$

Figure 1.3.4

The graph obtained from G by adding a set of edges E' is denoted by $G+E'$. If $E'=e,$ then we write $G-e$ for $G-\{e\}$ and $G+e$ for $G+\{e\}$.

Operation on graphs

Let G_1 and G_2 be subgraphs of G. We say that G_1 and G_2 are disjoint if they have no vertices in common. We say that they are edge disjoint if they have no edges in common.

The union of G_1 and G_2 is the subgraph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$. If G_1 and G_2 are disjoint, their union is also denoted by G_1+G_2.

If G_1 and G_2 have at least one vertex in common then their intersection is the subgraph with vertex set $V(G_1)\cap V(G_2)$ and edge set $E(G_1)\cap E(G_2)$.

The graph G and the union (intersection) of its subgraphs G_1 and G_2 are given in Figure 1.3.5.

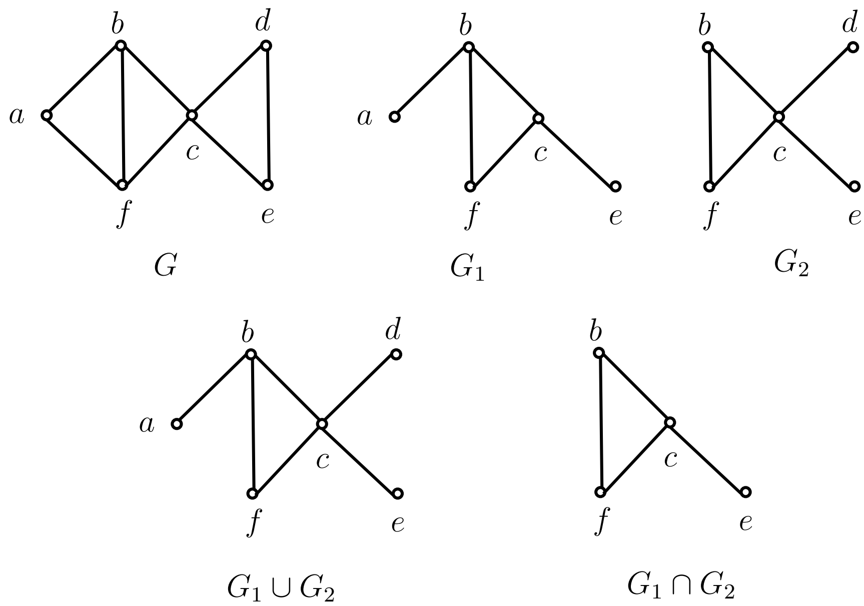


Figure 1.3.5

Definition 1.3.5. The *cartesian product* of two simple graphs G and H is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.
Graphs and their product are shown in Figure 1.3.6.

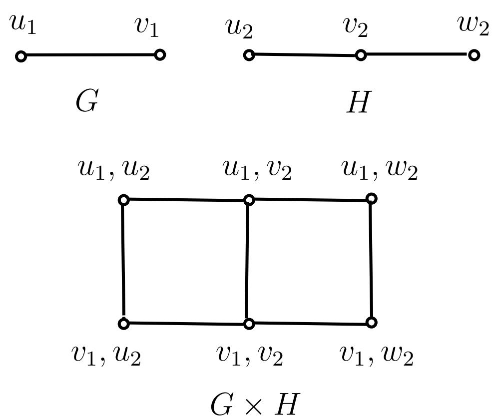


Figure 1.3.6

Definition 1.3.6. The *composition* of two simple graphs G and H is the simple graph with vertex set $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. It is denoted by $G[H]$.
The composition of two graphs G and H are given in Figure $1.3.7$.

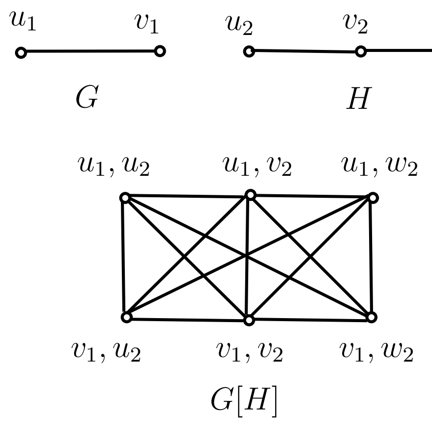


Figure 1.3.7

Exercises

1. Prove that every simple graph on n vertices is isomorphic to a subgraph of K_n.
2. Show that every induced subgraph of a complete graph is complete.
3. Show that every induced subgraph of a bipartite graph is bipartite.
4. Find a bipartite graph that is not isomorphic to a subgraph of any $k-$ cube.
5. Is $G[H] = H[G]$? Justify your assertion.

1.4 Degree sequences and Matrices

Definition 1.4.1. The degree of a vertex v in a graph G is the number of edges incident with $v,$ each loop counting as two. It is denoted by $d_G(v)$ or simply $d(v).$ The minimum degree of vertices of G is denoted by $\delta(G).$ The maximum degree of vertices of G is denoted by $\Delta(G).$

The following theorem is often called as **the fundamental theorem on graphs**.

Theorem 1.4.2. *The sum of the degrees of the vertices in any graph is twice the number of edges. That is,* $\sum\limits_{v\in V} d(v) = 2\epsilon.$

Proof. Every edge of G is incident with two vertices. Hence every edge contributes two to the sum of the degrees of the vertices.

Hence, $\sum\limits_{v\in V} d(v) = 2\epsilon.$ □

Corollary 1.4.3. *In any graph, the number of vertices of odd degree is even.*

Proof. Let V_1 denote the set of vertices of even degree; let V_2 denote the set of vertices

of odd degree. Then, $\sum\limits_{v\in V_1} d(v) + \sum\limits_{v\in V_2} d(v) = \sum\limits_{v\in V} d(v) = 2\epsilon,$ which is even.
Further, $d(v)$ is even for all $v \in V_1, \sum\limits_{v\in V_2} d(v)$ is even.

Hence, $\sum\limits_{v\in V_2} d(v)$ is even.

Since $d(v)$ is odd for all $v \in V_2,$ we have $|V_1|$ is even. $\square$

For the graph shown in Figure $1.4.1,$ $\delta(G) = 3$ and $\Delta(G) = 4.$

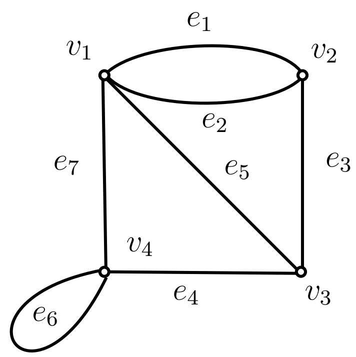


Figure $1.4.1$

Definition 1.4.4. A graph is said to be k *-regular* if $d(v) = k$ for all $v \in V(G).$ A regular graph is one that is k -regular for some $k.$ 3 -regular graphs are also known as *cubic graphs*.

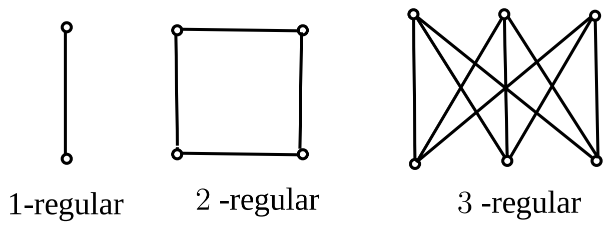
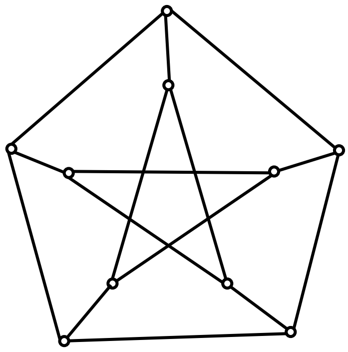


Figure $1.4.2$

Remark 1.4.5. $1.$ The complete graph K_n is regular of degree $n-1.$
$2.$ The complete bipartite graph $K_{n,n}$ is regular of degree $n.$
$3.$ The k -cube Q_k is regular of degree $k-1.$
$4.$ Peterson graph is 3 -regular and hence a cubic graph.



The Petersen Graph
Figure $1.4.3$

Definition 1.4.6. Let G be any graph with $V(G) = v_1, v_2, \cdots, v_\nu$. Then the sequence $d(v_1), d(v_2), \cdots, d(v_\nu)$ is called the *degree sequence* of G.

For example, the degree sequence of the graph in Figure $1.4.1$ is $(3, 3, 4, 4)$.

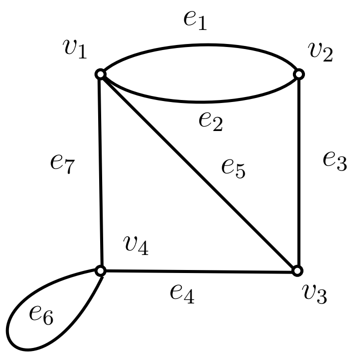


Figure $1.4.4$

Theorem 1.4.7. *A sequence $d(v_1), d(v_2), \cdots, d(v_\nu)$ of nonnegative integers is a degree sequence of G if and only if $\sum\limits_{i=1}^{\nu} d(v_i)$ is even.*

Proof. Assume that $d(v_1), d(v_2), \cdots, d(v_\nu)$, where $d_i \geq 0, 1 \leq i \leq \nu$ is the degree sequence of a graph G. Then by Theorem 1.4.2, $\sum\limits_{i=1}^{\nu} d(v_i) = 2\epsilon,$ which is even.

Conversely, assume that $d(v_1), d(v_2), \cdots, d(v_\nu)$ are nonnegative integers such that $\sum\limits_{i=1}^{\nu} d(v_i)$ is even. It is enough to construct a graph with vertex set v_i and $d(v_i) = d_i$ for all i. Since $\sum\limits_{i=1}^{\nu} d(v_i)$ is even, the number of odd integers is even. First form an arbitrary pairing of the vertices in $\{v_i \mid d(v_i) \; is \; even\}$ and join each pair by an edge. Now the remaining degree needed at each vertex is even, which can be obtained by adding $\left[\frac{d}{2}\right]$ loops at v_i.

$\square$

Definition 1.4.8. A sequence $D = (d_1, d_2, \cdots, d_n)$ is said to be *graphic* if there is a simple graph G with degree sequence D. Then G is called the realization of D. For example, the sequence $(4, 4, 2, 2, 1, 1)$ is graphic since it is the degree sequence of the graph G given below.

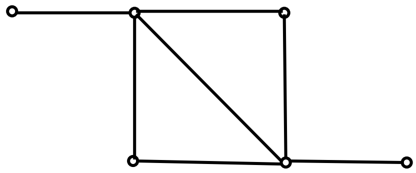


Figure 1.4.5

Theorem 1.4.9. *If* $d = (d_1, d_2, \cdots, d_n)$ *is graphic and* $d_1 \geq d_2 \geq \ldots d_n,$ *then* $\sum_{i=1}^{n} d(v_i)$ *is even and* $\sum_{i=1}^{n} d(v_i) \leq k(k-1) + \sum_{i=k+1}^{n} d(v_i)d_i$ *min*$\{k, d_i\}$ *for* $1 \leq k \leq n.$

Proof. Since d is graphic, it has a realization graph G. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $d(v_i) = d_i$. Then by Theorem 1.4.2,

$\sum_{i=1}^{\nu} d(v_i) = 2\epsilon,$ which is even.

$\sum_{i=1}^{\nu} d(v_i)$ is the sum of the degrees of the vertices $v_1, v_2, \cdots, v_n$.

It can be divided into two parts, the first part is the contribution to this sum by edges joining the vertices $v_1, v_2, \cdots, v_k$ and the second part is the contribution to this sum by edges joining one of the vertices $v_{k+1}, v_{k+2}, \cdots, v_n$.

Hence, $\sum_{i=1}^{n} d(v_i) \leq k(k-1) + \sum_{i=k+1}^{n} d(v_i)\, d_i \min\{k, d_i\}$ for $1 \leq k \leq n.$ $\square$

Solved problems

Problem 1. Find a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N},$ every graph of average degree at least $f(k)$ has a bipartite subgraph of minimum degree at least k.

Solution. Define a map $f : \mathbb{N} \to \mathbb{N}$ by $f(k) = 4k;\ \forall k \in \mathbb{N}$. The idea behind to consider this function is following: Every graph with an average degree of $4k$ have a subgraph H with minimum degree $2k,$ and we will lose another factor of 2 in moving H to its bipartite subgraph. Let $H^{'}$ be the bipartite subgraph of H with the maximal number of edges. My claim is that $H^{'}$ have minimum degree atleast $k.$ If not, let $v \in H^{'}$ such that $d_{H'}(v) < k:$ This means v lost more than half of its neighbours in the process to form H to $H^{'}.$ This means v is on the same partition with its looses neighbours. But in that case if we consider v in the other partition we can able to connect those previously looses vertices to v and form a new bipartite subgraph of H with more edges then $H^{'}$ have, a contradiction. Hence it proves of my claim. ■

Problem 2. Determine the order and the size of the hypercube $Q_k.$ Prove also that Q_k is k -regular and bipartite.

Solution. Clearly, $V(Q_k)$ is the set of all ordered k -tuples of $0'$ s and $1'$ s. Number of such tuples is $2^k.$ Therefore, $\nu(Q_k) = 2^k.$

Since two vertices are joined if and only if they differ in exactly one coordinate, it follows that each vertex is adjacent to exactly k vertices. Thus,

$$\begin{aligned}\epsilon(Q_k) &= \frac{k+k+\ldots+k \ (2^k \text{ times})}{2}, \text{ since each edge is incident with two vertices.}\\ &= k.\frac{2^k}{2}\\ &= k2^{k-1}\end{aligned}$$

Since two k -tuples form an edge if and only if they differ in exactly one position. Thus each vertex has degree k and so Q_k is k -regular.

Now, let $X = \{$k-tuples with even number of 0’s$\}$

$Y = \{$k-tuples with odd number of 0’s$\}$. Now,

$X \cup Y = Q_k$ and $X \cap Y = \phi$

Also, any two vertices of $X($or $Y)$ differ at least in two coordinates and hence they are not adjacent. Thus any edge must have one end in X and the other end in $Y.$ Thus (X, Y) is a bipartition of $Q_k,$ which completes the proof. ■

Problem 3. Prove that $\delta \leq 2 \ \frac{\epsilon}{\nu} \ \leq \Delta.$

Solution. For any vertex v in any graph $G,$ $\delta(G) \leq d(v) \leq \Delta(G).$

Taking the sum over all the vertices of $V,$ we get

$|V|\delta(G) \leq \sum_{v\in V} d(v) \leq |V|\Delta(G).$

$\Rightarrow \nu\delta \leq 2\epsilon \leq \Delta\nu$

Dividing by $\nu,$ we get $\delta \leq 2\ \frac{\epsilon}{\nu}\ \leq \Delta.$ ■

Problem 4. If a $k-$ regular bipartite graph with $k > 0$ has bi-partition $(X, Y),$ prove that $|X| = |Y|.$

Solution Let G be a $k-$ regular bipartite graph with $k > 0.$ Since G is bipartite, every edge has one end in X and another end in $Y.$

Hence the number of edges incident with the vertices of X is equal to the number of edges incident with the vertices of $Y.$ Therefore,

$k.|X| = k.|Y|,$ since each vertex is of degree $k.$

$\Rightarrow |X| = |Y|,$ since $k > 0.$ ■

Problem 5. In any group of two or more people, prove that there are always two with the same number of friends.

Solution We construct a graph G by taking the group of n people as the set of vertices and joining two of them if they are friends. Then $d(v) =$ number of friends of v and hence we need only to prove that at least two vertices of G have the same degree.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}.$ Clearly $0 \leq d(v_i) \leq n-1$ for each $i.$

Suppose no two vertices of G have the same degree. Then the degrees of $v_1, v_2, \ldots, v_n$ are the integers $0, 1, 2, \ldots,\ n-1$ in some order. However a vertex of degree $n-1$ is joined to every other vertex of G and hence no point can have degree 0, which is a contradiction.

Hence there exist two vertices of G with equal degree. ■

Problem 6. Prove that the sequence $(7, 6, 5, 4, 3, 3, 2)$ is not graphic.

Solution. Let $d = (7, 6, 5, 4, 3, 3, 2).$

Suppose d is graphic. Let G be a realization of $d.$

Since there are 7 digits in the sequence, G has seven vertices and hence the maximum degree in G cannot exceed 7-1=6.

This contradicts the first digit in d.

Hence the given sequence is not graphic. ■

Problem 7. Prove that the sequence $(6, 6, 5, 4, 3, 3, 1)$ is not graphic.

Solution. Let $d = (6, 6, 5, 4, 3, 3, 1)$.

Suppose d is graphic. Let G be a realization of d.

Since there are 7 digits in the sequence, G has seven vertices.

The first two digits of d shows that there are two vertices which are adjacent to all the remaining 6 vertices.

Thus every vertex is adjacent to these two vertices and hence every vertex is of degree at least two.

This contradicts the last digit in d.

Hence the given sequence is not graphic. ■

Matrices of a graph

We study about two representations of a graph in matrix form. A matrix is a convenient and useful way of representing a graph to a computer. Further the algebra of matrices can be used to identify certain properties of graphs.

Definition 1.4.10. Let $G = (V(G), E(G))$ be a graph with $V(G) = \{v_1, v_2, \cdots v_\nu\}$ and $E(G) = \{e_1, e_2, \cdots e_\epsilon\}$. Then the *incidence matrix* of G is the $\nu \times \epsilon$ matrix defined by $M(G) = [m_{ij}]$, where m_{ij} is the number of times $(0,\ 1\ or\ 2)$ that v_i and e_j are incident.

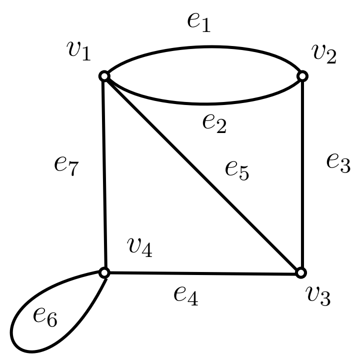


Figure 1.4.6

The incidence matrix of the above graph is as follows:

$$\text{M(G)=}\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \end{array}$$

Remark 1.4.11. $1.$ Since each edge is incident with exactly two vertices, each column sum of M is $2.$
$2.$ Sum of the i th row of M is equal to the degree of $v_i.$
$3.$ If G is simple, then the matrix M is a binary matrix with $0'$ s and $1'$ s.

Definition 1.4.12. Let $G = (V(G), E(G))$ be a graph with $V(G) = v_1, v_2, \cdots v_\nu$. Then the adjacency matrix of G is the $\nu \times \nu$ matrix defined by
$A(G) = [a_{ij}],$ where a_{ij} is the number of edges joining v_i and $v_j.$

The incidence matrix of the graph G shown in Figure 1.4.6 is as follows:

$$\text{A(G)=}\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Remark 1.4.13. $1.$ The adjacency matrix $A(G)$ is symmetric.
$2.$ If G is simple, then the entries along the principal diagonal are zero.
$3.$ The sum of the i^{th} row (column) of $A(G)$ is equal to the degree of $v_i.$

Exercises

1. Find the degrees of the vertices of the graph G given in Figure 1.3.1.
2. Find the incidence matrix M and adjacency matrix A of the graph given in Figure 1.3.1.
3. If G is simple, prove that the entries on the diagonals of both MM' and A^2 are the degrees of the vertices of $G.$

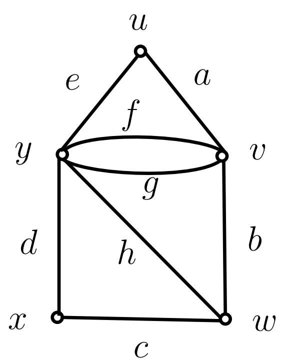
1.5 Paths and Connection

Definition 1.5.1. A *walk* in G is a finite nonnull sequence $W = v_0e_1v_1e_2\ldots e_kv_k,$ whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k,$ the ends of q_i are v_{i-1} and $v_i.$

We say that W is a walk from v_0 to $v_k,$ or a (v_0, v_k) - walk. The vertices v_0 and v_k are called the *origin* and *terminus* of $W,$ respectively and $v_1, v_2, \ldots, v_{k-1}$ its *internal vertices*. The integer k is the length of $W.$

If $W = v_0e_1v_1e_2\ldots e_kv_k$ and $W' = e_kv_{k+1}e_{k+1}\ldots e_lv_l$ are walks, the walk $v_ke_kv_{k-1}\ldots e_1v_0,$ obtained by reversing $W,$ is denoted by W^{-1} and the walk $v_0e_1v_1\ldots e_kv_ke_{k+1}v_{k+1}\ldots e_1v_1$ obtained by concatenating W and W' at v_k is denoted by $WW'.$

Definition 1.5.2. A *section* of a walk $W = v_0e_1v_1e_2\ldots e_kv_k$ is a walk that is a subsequence $v_ie_{i+1}v_{i+1}\ldots e_jv_j$ of consecutive terms of $W;$ we refer to this subsequence as the (v_i, v_j) - section of $W.$



Walk : $uavfvfvgyhwbv$

Trail : $wcxdyhwbvgy$

Path : $xcwhyeuav$

Figure 1.5.1

Remark 1.5.3. In a simple graph, a walk $v_0e_1v_1e_2\ldots e_kv_k$ is determined by the sequence $v_0v_1\ldots v_k$ of its vertices; hence a walk in a simple graph can be specified simply by its vertex sequence.

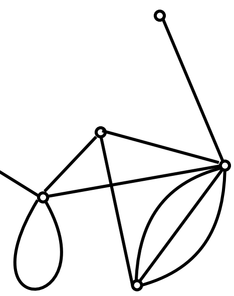
Moreover, even in graphs that are simple, we shall sometimes refer to a sequence of vertices in which consecutive terms are adjacent as a 'walk'.
In such cases, it should be understood that the discussion is valid for every walk with that vertex sequence.

If the edges $e_1, e_2, \ldots, e_k$ of a walk W are distinct, W is called a *trail*. In this case the length of W is just $\epsilon(W).$ If, in addition, the vertices $v_0, v_1, \ldots, v_k$ are distinct, W is called a path.

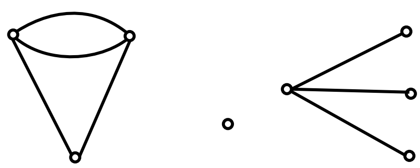
We shall also use the word *path* to denote a graph or a subgraph whose vertices and edges are the terms of a path.

Definition 1.5.4. Two vertices u and v of G are said to be *connected* if there is a (u, v) - path in $G.$ Connection is an equivalence relation on the vertex set $V.$ Thus, there is a partition of V into nonempty subsets $V_1, V_2, \ldots, V_\omega$ such that two vertices u and v are connected if and only if both u and v belong to the same set $V_i.$ The subgraphs $G[V_1], G[V_2], \ldots, G[V_\omega]$ are called the *components* of $G.$ If G has exactly one component, G is connected; otherwise, G is disconnected. We denote the number of components of G by $\omega(G).$

Connected and disconnected graphs are shown below:



(a) A connected graph

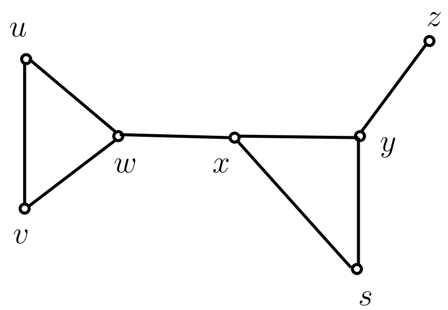


(b) A disconnectd graph with three components

Figure $1.5.2$

Definition 1.5.5. If two vertices u and v are connected in a graph $G,$ then the *distance* between them is defined to be the length of a shortest $(u, v)-$ path in $G.$ It is denoted by $d(u, v).$ If there is no path connecting u and $v,$ then $d(u, v)$ is defined to be infinite. For the graph given below,

$$d(u, w) = 1 \qquad d(x, y) = 1$$
$$d(u, x) = 2 \qquad d(x, z) = 3$$
$$d(u, y) = 3 \qquad d(u, z) = 4$$



Distance between vertices

Figure $1.5.3$

Definition 1.5.6. The *diameter* of a graph G is defined to be the maximum distance between two vertices of G. It is denoted by $diam(G)$.
The diameter of the graph given in Figure $1.5.3$ is $max\ \{1,2,3,4\} = 4$.

Definition 1.5.7. If v is a vertex of a graph G, then the *eccentricity*, denoted by $e(v)$, is defined by

$e(v) = max\{d(u,v) : u \in V(G)\}$

Definition 1.5.8. The *radius* of G is the minimum eccentricity of G. It is denoted by $r(G)$.

That is, $r(G) = min\{e(v) : v \in V(G)\}$

A vertex V is called a *central vertex* if $e(v) = r(G)$. The set of all central vertices of G is called the *center* of G.

For the graph given in Figure $1.5.3$,

$e(u) = 4,\quad e(x) = 2,\quad e(w) = 3,\quad e(z) = 4,$

$e(y) = 3,\quad e(v) = 4,\quad e(s) = 3.$

$r(G) = min\ \{2,3,4\} = 2.$

$Centre = \{x\}.$

Definition 1.5.9. A walk is *closed* if it has positive length and its origin and terminus are the same. A closed trail whose origin and internal vertices are distinct is called a cycle. A cycle of length k is called a k *-cycle*. A cycle is odd or even according as k is odd or even. A 3 -cycle is called as *triangle*. The length of the shortest cycle is called the *girth* of the graph.

A characterization of bipartite graphs

Theorem 1.5.10. *A graph is bipartite if and only if it contains no odd cycle.*

Proof. Let G be a bipartite graph with bipartition (X, Y).
To prove that G contains no odd cycle, it is enough to prove that every cycle in G is of even length.

Let $C = v_0 v_1 \dots v_k v_0$ be any cycle of length $k+1$ in G. Without loss of generality, we may assume that $v_0 \in X$. Since $v_0 v_1 \in E(G)$ and G is bipartite, we have $v_1 \in Y$. Similarly $v_2 \in X$. In general, $v_{2i} \in X$ and $v_{2i+1} \in Y$. Since $v_0 \in X, v_k \in Y$. Thus $k = 2i+1$ for some i and it follows that $k+1$ is even.

Conversely, let G be a connected graph with no odd cycle.
We choose an arbitrary vertex u and define a partition (X, Y) of V by setting

$X = \{x \mid d(u, x) \text{ is even}\}$ and

$Y = \{y \mid d(u, y) \text{ is odd}\}$.

Now let us show that (X, Y) is a bipartition of G.

It is enough to prove that no two vertices in X as well as in Y are adjacent in G. Let v and w be two vertices of X. Let P be the shortest (u, v)-path, let Q be the shortest (u, w)-path. Let u' denote the last vertex common to both P and Q. (See Figure $1.5.4$; where dark lines denotes the path P and the thin line denotes the path Q)

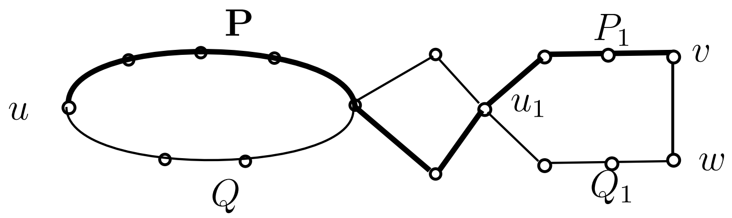


Figure $1.5.4$

Since P and Q are the shortest paths, the $(u, u')-$ sections of both P and Q are the shortest $(u, u')-$ paths and hence, have the same length. Since the lengths of both P and Q are even, the lengths of the $(u', v)-$ section P_1 of P and the $(u', w)-$ section Q_1 of Q must have the same parity. It follows that the $(v, w)-$ path $P_1^{-1} Q_1$ is of even length. If v were joined to $w, P_1^{-1} Q_1 w v$ would be a cycle of odd length, contradiction to the hypothesis. Therefore, no two vertices in X are adjacent. Similarly, we can prove that no two vertices in Y are adjacent in G. Hence G is a bipartite graph. $\square$

Solved Problems

Problem 1. Show that $rad(G) \leq diam(G) \leq 2\, rad(G)$.

Solution. We know that $diam(G) = \max_{x,y\in V(G)} d_G(x,y)$.

$$\begin{aligned} rad(G) &= \min_{x\in V(G)} \max_{y\in V(G)} d_G(x,y) \\ &\leq \min_{x\in V(G)} \max_{y\in V(G)} diam(G) \\ &= diam(G). \end{aligned}$$

To show $diam(G) \leq 2\, rad(G)$.

Let $a,b,v \in V(G)$ such that $d_G(a,b) = diam(G)$ and $rad(G) = \max_{y\in V(G)} d_G(v,y)$.

$$\begin{aligned} diam(G) &= d_G(a,b) \leq d_G(a,v) + d_G(v,b). \\ &\leq rad(G) + rad(G) = 2\, rad(G). \end{aligned}$$

■

Problem 2. If there is a (u,v) -walk in G, prove that there is also a (u,v) -path in G.

Solution. We prove the result by induction on the length of the walk.

Any walk of length 0 or 1 is obviously a path. Therefore the result is true if the length of the given walk is 0 or 1.

Assume that the result is true for all walks of length at most $k-1$.

Let $W : u = u_0, u_1, \ldots, u_k = v$ be a (u,v) -walk of length k. If all the vertices of W are distinct, then it is obviously a path. If not, there exist i and j such that $0 \leq i < j \leq k$ and $u_i = u_j$.

Then $W' : u = u_0, u_1, \ldots, u_i, u_{j+1}, \ldots, u_k = v$ is a (u,v) -walk of length at most $k-1$ in G. So, by induction assumption, the walk W' and hence W contains a (u,v) -path. ■

Problem 3. Show that the number of (v_i, v_j) -walks of length k in G is the (i,j) th entry in A^k, where A is the adjacency matrix of G.

Solution. We prove the result by induction on k.

The adjacency matrix of G is the $\nu \times \nu$ matrix

$A = [a_{ij}]$, where a_{ij} is the number of edges joining v_i and v_j.

The number of (v_i, v_j) -walks of length one =

$$\begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

$$= a_{ij}$$

Hence the result is true for $k = 1.$

We now assume that the result is true for $k - 1.$

Let $A^{k-1} = [a_{ij}^{(k-1)}],$ where $a_{ij}^{(k-1)}$ is the number of (v_i, v_j) -walks of length $k - 1.$

$A^{k-1}A = [a_{ij}^{(k-1)}](a_{ij})$

Hence, the $(i, j)^{\text{th}}$ entry of $A_k = \sum_{r=1}^{\nu} a_{ir}^{(k-1)} a_{rj}$

Also every (v_i, v_j) -walk of length k in G consists of a (v_i, v_r) -walk of length $k - 1$ followed by a vertex v_j which is adjacent to $v_k.$ Hence if v_j is adjacent to $v_k,$ then $a_{ir}^{(k-1)} a_{rj}$ represents the number of (v_i, v_j) -walks of length k in $G.$

This completes the induction and the proof. ■

Problem 4. If G is simple and $\delta \leq k,$ prove that G has a path of length $k.$

Solution. Let $P = v_0, v_1, \ldots, v_r$ be a longest path in $G.$ Then the vertices adjacent to v_r in G can only be from $v_0, v_1, \ldots, v_{r-1}$ as otherwise we would get a path of length larger than $r,$ giving a contradiction.

Hence $r \geq d(v_r) \geq \delta \geq k.$ Thus P has length at least k and hence G has a path of length k (namely either P or its section). ■

Problem 5. If G is disconnected, then prove that G^c is connected.

Solution. Let u and v be two vertices of G^c (and therefore of G). If u and v belong to different components of $G,$ then obviously u and v are nonadjacent in G and so they are adjacent in $G^c.$ Thus u and v are connected (by a path) in $G^c.$ In case u and v belong to same component of $G,$ choose a vertex w of G not belonging to this component of G (this is possible because G has at laest two components). Then uw and vw are not edges of G and hence they are edges of $G^c.$ Then u, w, v is a (u, v) -path in $G^c.$ Thus G^c is connected. ■

Problem 6. If $e \in E(G),$ prove that $\omega(G) \leq \omega(G - e) \leq \omega(G) + 1.$

Solution. Since the deletion of an edge does not affect the connectedness of other components, it is enough if we prove the result for a connected graph.

Let G be a connected graph. Then $\omega(G) = 1.$

We have to prove that $1 \leq \omega(G - e) \leq 2.$

Consider $G - e$ where $e = uv.$ Let w be any vertex of $G.$

If w is adjacent to both u and v in $G - e,$ then any two vertices are connected in $G - e$ and hence $\omega(G - e) = 1.$

Otherwise, let V_1 denote the set of vertices which are connected to u in $G - e$ and V_2

denote the set of vertices which are connected to v in $G-e$. Then the induced subgraphs $G[V_1]$ and $G[V_2]$ are the two components of $G-e$ and hence $\omega(G-e)=2$. ■

Problem 7. Prove that any two longest paths in a connected graph have a vertex in common.

Solution. Suppose $P=u_1,u_2,\ldots,u_k$ and $Q=v_1,v_2,\ldots,v_k$ are two longest paths in G having no vertex in common. As G is connected, there exists a u_1-v_1 path P' in G. Certainly, there exist vertices u_r and v_s of $P', 1\le r\le k, 1\le s\le k$ such that the (u_r,v_s) -section P'' of the path P' has no internal vertex in common with P or Q.

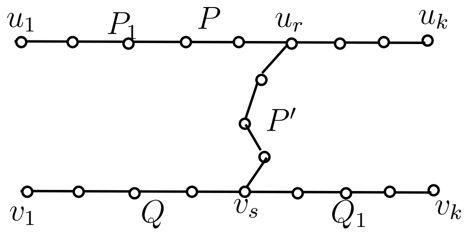


Figure 1.5.5

Now, of the two sections u_1-u_r and u_r-u_k of P, one must have length at least $\frac{k}{2}$. Similarly, of the two sections v_1-v_s and v_s-v_k of Q, one must have length at least $\frac{k}{2}$. Let these sections be P_1 and Q_1 respectively. Then $P_1\cup P''\cup Q_1$ is a path of length at least $\frac{k}{2}+\frac{k}{2}+1$, contradicting k being the length of a longest path in G. ■

Problem 8. If G is simple and connected but not complete, prove that G has three vertices u,v and w such that $uv,vw\in E(G)$ and $uw\notin E(G)$.

Solution. Since G is not complete, there are two nonadjacent vertices, say v_0 and v_k in G. Since G is connected, there exists a path joining v_0 and v_k in G. Let $P=v_0,v_1,\ldots,v_k$ be a shortest (v_0,v_k) -path in G. Since $v_0v_k\notin E(G)$, it follows that $k\ge 2$ and also we have v_k is not adjacent to v_{k-2}. Then $u=v_{k-2}, v=v_{k-1}$ and $w=v_k$ are the three vetices such that $uv,vw\in E(G)$ but $uw\notin E(G)$. ■

Problem 9. If G is simple graph of order n and $\delta\ge\frac{n-1}{2}$, prove that G is connected.

Solution. Assume, to the contrary, that G has at least two components, say G_1 and G_2. Let v be any vertex of G_1. As $\delta\ge\frac{(n-1)}{2}, d(v)\ge\frac{(n-1)}{2}$. Since v and all its neighbours are confined into a single component, the component G_1 contains at least $d(v)+1\ge\frac{(n-1)}{2}+1=\frac{(n+1)}{2}$ vertices. Similarly, G_2 contains at least $\frac{(n+1)}{2}$ vertices. Therefore, G has at least $\frac{(n+1)}{2}+\frac{(n+1)}{2}=n+1$ vertices, which is a contradiction. ■

Exercises

1. Prove that G is connected if and only if, every partition of V into nonempty sets V_1 and $V_2,$ there is an edge with one end in V_1 and another end in $V_2.$
2. If G is simple and $\epsilon > \binom{\nu-1}{2},$ show that G is connected.
3. For $v > 1,$ find a disconnected simple graph G with $\epsilon = \binom{\nu-1}{2}.$
4. If G is simple and $\delta > [\frac{\nu}{2}] - 1,$ show that G is connected.
5. Find a disconnected $([\frac{\nu}{2}] - 1)$ -regular simple graph of even order.
6. If G is connected and each degree in G is even, prove that $\epsilon(G - v) \leq \frac{d(v)}{2}$ for any $v \notin V.$
7. Prove that the distance d between two vertices is a metric on $V.$
8. If G has diameter greater than three, prove that G^c has diameter less than three.
9. If G is simple with diameter two and $\Delta = v - 2,$ prove that $\epsilon \leq 2\nu - 4.$

1.6 Trees

Definition 1.6.1. A graph containing no cycles is called an *acyclic graph*. A tree is a connected acyclic graph.

All the nonisomorphic trees on six vertices are given in Figure 1.6.1.

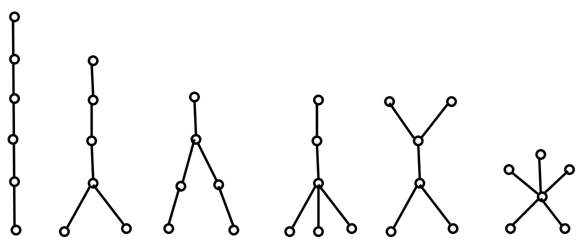


Figure $1.6.1$ Nonisomorphic trees of order 6

Theorem 1.6.2. *In a tree, any two vertices are connected by a path.*

Proof. Assume, to the contrary, that there are two distinct (u, v) -paths, say P_1 and P_2.

Then there exists an edge $e = xy$ of P_1 that is not an edge of P_2. Clearly, the graph $(P_1 \cup P_2) - e$ is connected. Therefore it contains an (x, y) -path, say P.

Now, $P + e$ forms a cycle in G, which is a contradiction to the hypothesis that G is a tree. □

Remark 1.6.3. The converse of the above theorem holds for graphs with no loops.

Proof. Let G be a connected graph with no loops such that any two vertices are connected by an unique path.

We have to prove that G is acyclic.

If possible, suppose G contains a cycle, say C of length k. Let $e = uv \in E(C)$. Then the edge uv itselt is a (u, v) -path of length 1 and $C - e$ is a (u, v) - path of length $k - 1$ in G. Thus, the vertices u and v are connected by two distinct paths, giving a contradiction.

□

Remark 1.6.4. The number of edges for all the trees in Figure $1.6.1$ is 5. The following theorem provides a proof for this.

Theorem 1.6.5. *If G is a tree, then $\epsilon = \nu - 1$.*

Proof. We prove the result by induction on ν.

If $\nu = 1$, then $G \cong K_1$ and $\epsilon = 0 = \nu - 1$.

Assume that the theorem is true for all trees on fewer than ν vertices.

Let G be a tree on $n \geq 2$ vertices. Let $uv \in E$. Then $G - uv$ contains no $(u, v)-$ path, since uv is the unique (u, v) path in G. Thus $G - uv$ is disconnected and so $\omega(G - uv) = 2$.

Let G_1 and G_2 be the two components of $G - uv$. Since G_1 and G_2 are subgraphs of the tree G, both are acyclic and hence trees.

Moreover, each tree G_i has fewer than ν vertices. Therefore, by induction assumption,

$$\epsilon(G_i) = \nu(G_i) - 1 \text{ for } i = 1, 2$$

Thus,
$$\begin{aligned} \epsilon(G) &= \epsilon(G_1) + \epsilon(G_2) + 1 \\ &= \nu(G_1) - 1 + \nu(G_2) - 1 + 1 \\ &= \nu(G_1) + \nu(G_2) - 1 \\ &= \nu(G) - 1 \end{aligned}$$

Hence the proof. □

Corollary 1.6.6. *Every nontrivial tree has at least two end vertices.*

Proof. Let G be a nontrivial tree. Then $d(v) \geq 1, \forall\, v \in V$.
We know that $\sum d(v) = 2\epsilon$

$$= 2(\nu - 1)$$
$$= 2\nu - 2 \qquad \ldots \text{ (Eq 1)}$$

If no vertex has degree 1, then $\delta \geq 2$ and hence

$\sum d(v) \geq 2\nu,$ which contradicts (Eq 1)

If G has only one vertex of degree 1, then $d(v) \geq 2$ for $\nu - 1$ vertices and hence

$\sum d(v) \geq 2(\nu - 1) + 1$

$2\nu - 2 + 1$

$= 2\nu - 1,$ which again contradicts (Eq 1).

Therefore, $d(v) = 1$ for at least two vertices. □

Definition 1.6.7. An acyclic graph is called a forest. Each component of a forest is a tree. Figure 1.6.2 illustrates a forest.

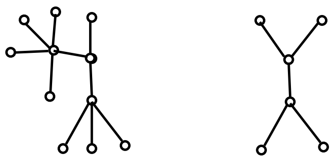


Figure 1.6.2. A forest

Remark 1.6.8. If G is a forest, then $\epsilon = \nu - \omega,$ where ω is the number of components of G.

Solved Problems

Problem 1. Prove that every tree with exactly two vertices of degree one is a path.

Solution. Let T be a tree with exactly two vertices of degree 1. We have to prove that T is a path.

Suppose T is not a path. Then T has at least one vertex of degree 3. Let it be u. Consider the maximal sub trees in which u is a vertex of degree 1. Then we have at least three maximal subtrees. Since each such subtree has at least two vertices of degree 1, we have three subtrees each with at least one vertex of degree 1 other than u. These vertices of degree 1 are also vertices of degree 1 in T. Thus T has at least three vertices of degree 1, which is a contradiction.

Hence T is a path. ■

Problem 2. If G is a graph with $\nu - 1$ vertices, prove that the following are equivalent.

(a) G is connected

(b) G is acyclic
(c) G is a tree
Solution.
$(a) \Rightarrow (b)$ Let G be connected. We have to prove that G is acyclic. Suppose G has a cycle, say $C.$ Let e be an edge on $C.$ Then $C-e$ is connected. Delete all the edges on the cycle successively such that the resulting graph remains connected but has no cycles. Thus we get a connected acyclic graph T (tree) on ν vertices. Hence

$$\begin{aligned}\epsilon(G) &\geq \text{(number of edges of } T\text{)+1}\\ &= (\nu-1)+1\\ &= \nu\end{aligned}$$

This is a contradiction and hence G has no cycles.
$b) \Rightarrow c)$ Assume that G is acyclic. We have to prove that G is connected. Suppose not. Then it has $k(\geq 2)$ components say $G_1, G_2, \ldots, G_k.$ Since G is acyclic, each component is acyclic and connected. Thus each component is a tree and hence by Theorem 1.6.5,

$$\epsilon(G_i) = \nu(G_i) - 1 \text{ for each } i = 1, 2, \ldots, k.$$

Hence,
$$\begin{aligned}\epsilon(G) &= \epsilon(G_1) + \epsilon(G_2) + \ldots + \epsilon(G_k) - 1\\ &= \nu(G_1) - 1 + \nu(G_2) - 1 + \ldots + \nu(G_k)\\ &= \nu(G_1) + \nu(G_2) + \ldots + \nu(G_k) - k\\ &= \nu(G) - k\\ &< \nu - 1, \text{ since } k \geq 2.\end{aligned}$$

This is a contradiction. Hence G is connected and so it is a tree.
(c) $\Rightarrow$ (a) is obvious. ■

Problem 3. If G is a tree with $\Delta \geq k,$ prove that G has at least k end vertices.
Solution. Let G be a tree; let u be a vertex of degree $\Delta \geq k.$
Consider the maximal subtrees in which u occurs as an endverex. Then we have at least k such subtrees. Since each such subtree has at least one end vertex other than u and the end vertex is also an end vertex in $G,$ it follows that G has at least k end vertices.■

Recall that the centre of G is the set of all vertices of minimum eccentricity.
Problem 4. Prove that the centre of a tree consists of either one vertex or two adjacent vertices.
solution The result is obvious for the trees K_1 and $K_2;$ the vertices of K_1 and K_2 are central vertices.
Now let T be a tree with $\nu(T) \geq 3.$ Then, by Corollary 1.6.6, T has at least two end vertices. Clearly, the end vertices of T cannot be the central vertices.

Delete all the end vertices from T. This result in a subtree T' of T. Since any path of maximum length in T starting from any vertex of T' will end at an end vertex of T, the eccentricity of each vertex of T' is one less than that in T.
Hence, the vertices of minimum eccentricity in T' are same as those in T. In other words, T and T' have the same centre.
Similarly, if T'' is the tree obtained from T' by deleting all its end vertices, then T'' and T' have the same centre.
Repeat this process of deleting the end vertices from the successive subtrees until these results in a K_1 or K_2. This will always be the case as T is finite.
Hence, the centre of T is either a single vertex or a pair of adjacent vertices. ■

The process of determining the centre of a tree as described above is illustrated in Figure $1.6.3$.

T :
T' :
T'' :
T''' :

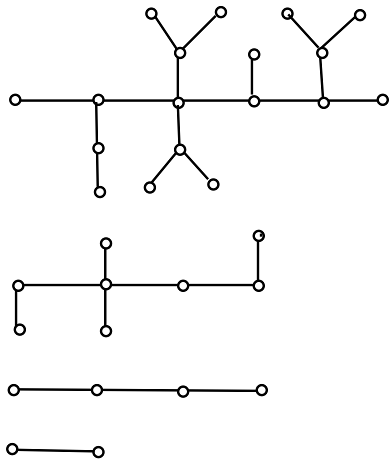


Figure $1.6.3$. Process of determining the centre of T

Problem 5. Prove that the sequence $(d_1, d_2, \ldots, d_v)$ of positive integers is the degree sequence of a tree if and only if $\sum i = 1^\nu d_i = 2(\nu - 1)$.
Solution. The solution of the problem is trivial if $\nu = 1$. So, we can assume that $\nu \geq 2$.
Necessity: Assume that the sequence $(d_1, d_2, \ldots, d_v)$ of positive integers is the degree sequence of a tree, say T. Since T is connected and nontrivial, it has no isolated vetices. Hence every term of the degree sequence is positive.
Therefore, by Theorems 1.4.2 and 1.6.5, $\sum\limits_{i=1}^{\nu} d(v_i) = 2\epsilon = 2(\nu - 1)$.

Conversely, assume that the sequence $D = (d_1, d_2, \ldots, d_v)$ of positive integers, where $\sum d_i = 2(\nu - 1)$, is the degree sequence of a graph G.
To prove G is a tree. We proceed by induction on ν (≥ 2).

If $\nu = 2,$ then we have $d_1 + d_2 = 2(2-1) = 2.$ Since $d_1 \geq 1, d_2 \geq 1,$ we get $d_1 = 1$ and $d_2 = 1.$ Then the unique realization of G is $K_2,$ which is clearly a tree.

Assume that if the sequence $D = (d_1, d_2, \ldots, d_n)$ of positive integers, where $\sum\limits_{i=1}^{\nu} d_i = 2(n-1),$ is the degree sequence of a graph $H,$ then H is a tree, where $2 \leq n \leq \nu - 1.$

Let the sequence $D = (d_1, d_2, \ldots, d_v)$ of positive integers, where $\sum\limits_{i=1}^{\nu} d_i = 2(\nu - 1),$ is the degree sequence of a graph $G.$ If $d_i \geq 2,$ for every $i, 1 \leq i \leq \nu,$

then $\sum\limits_{i=1}^{\nu} d_i \geq 2\nu > 2(\nu - 1),$ giving a contradiction.

Hence $d_i < 2$ for at least one $i.$ For this $i,$ $d_i = 1$ because $d_i \geq 1 \; \forall \, i.$ For definiteness, let us assume that $d_\nu = 1$ and let v_ν be the corresponding vertex. Let v_k be the unique vertex adjacent to v_ν in G with $d_G(v_k) = d_k.$

Consider the subgraph $G' = G - v_\nu$ and the sequence $(d_1, \ldots, d_{k-1}, d_k - 1, d_{k+1}, \ldots, d_{\nu-1}).$ Let $v_1, v_2, \ldots, v_{\nu-1}$ be the respective vertices with these degrees in order in $G'.$ Note that d_k cannot be equal to $1,$ since in that case the edge $v_k v_\nu$ itself forms a separate component of $G,$ a contradiction to G is connected and $\nu \geq 3.$ Thus, the degree sequence of G' has positive terms.

Further, $d_1 + \ldots + d_{k-1} + d_k - 1 + d_{k+1} + \ldots + d_{\nu-1}$

$$
\begin{aligned}
&= (d_1 + \ldots + d_{\nu-1}) - 1 \\
&= (\textstyle\sum_{i=1}^{\nu} d_i) - d_\nu - 1 \\
&= 2(\nu - 1) - 1 - 1 \\
&= 2((\nu - 1) - 1).
\end{aligned}
$$

By induction assumption, G' is a tree. Now, to realize G from $G',$ attach the pendant edge $v_k v_\nu$ at $v_k.$ Therefore, G is a tree. ■

Exercises

1. If G is a forest with exactly $2k$ vertices of odd degree, prove that there are k edge disjoint paths $P_1, P_2, \ldots, P_k$ in G such that $E(G) = E(P_1) \cup E(P_2) \cup \ldots \cup E(P_k).$

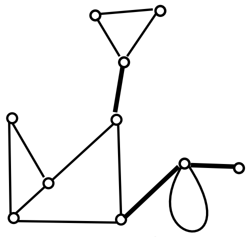
2. Let T be an arbitrary tree on $k+1$ vertices. If G is simple and $\delta \geq k,$ prove that G has a subgraph isomorphic to $T.$

1.7 Cut Edges and Cut Vertices

Cut edges and Bonds

Definition 1.7.1. A cut edge of G is an edge e such that $\omega(G-e) > \omega(G)$.

The dark edges of the graph shown in Figure $1.7.1$ are cut edges.



Cut edges of a graph
Figure $1.7.1$

Theorem 1.7.2. *An edge e of G is a cut edge of G if and only if e is contained in no cycle of G.*

Proof. Let e be a cut edge of G. We have to prove that e is contained in no cycle of G.

Since $\omega(G-e) > \omega(G)$, there exist vertices u and v of G that are connected in G but not in $G-e$. Therefore, there is some $(u,v)-$ path P in G which necessarily traverse e. Let x and y be the ends of e and assume that x precedes y in P. In $G-e, u$ is connected to x by a section of P and y is connected to v by a section of P.

Suppose e were in a cycle, say C. Then x and y would be connected in $G-e$ by the path $C-e$. Thus u and v would be connected in $G-e$, which is a contradiction.

Conversely, assume that $e = xy$ is contained in no cycle of G. We have to prove that e is a cut edge of G. Suppose e is not a cut edge of G. Then $\omega(G-e) = \omega(G)$. Since there is an $(x,y)-$ path (namely the edge xy) in G, vertices x and y are in the same component of G. It follows that x and y are in the same component of $G-e$ and hence there is a $(x,y)-$ path P in $G-e$. But then $P+e$ is a cycle of G containing the edge e, which is a contradiction. Hence e is a cut edge of G. □

Theorem 1.7.3. *An edge e is a cut edge of a connected graph G if and only if there exist vertices u and v such that e belongs to every $(u,v)-$ path.*

Proof. Let $e = xy$ be a cut edge of G. Then $G - e$ has two components, say G_1 and G_2. Let u be in G_1 and v be in G_2. Then clearly every $(u, v)-$ path in G contains e.

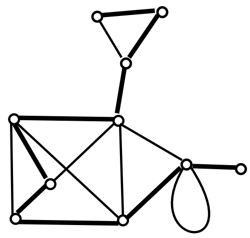
Conversely, assume that there exist two vertices u and v such that e belongs to every $(u, v)-$ path in G. Then there exists no $(u, v)-$ path in $G - e$. Hence $G - e$ is disconnected and so e is a cut edge of G. □

Theorem 1.7.4. *A connected graph is a tree if and only if every edge is a cut edge.*

Proof. Let G be a tree and e be an edge of G.
Since G is acyclic, the edge e is contained in no cycle of G.
Therefore, by Theorem 1.7.3, the edge e is a cut edge of G.
Conversely, assume that every edge of a connected graph G is a cut edge.
Suppose that G is not a tree.
Then G contains a cycle.
Therefore, by Theorem 1.7.3, the edges in the cycles of G are not cut edges of G, which is a contradiction.
Therefore, G is a tree. □

Definition 1.7.5. A spanning tree of G is a spanning subgraph of G that is a tree.

For example, a spanning tree of the graph G in Figure 1.7.2 is indicated by the dark edges.



A spanning tree of a graph
Figure 1.7.2

Corollary 1.7.6. *Every connected graph contains a spanning tree.*

Proof. Let G be connected and let T be a minimal connected spanning subgraph of G (T exists because G is a connected spanning subgraph of itself).
By definition, $\omega(T) = 1$ and $\omega(T - e) > 1$ for each edge e of T.
Therefore each edge of T is a cut edge. Therefore T is a tree by Theorem 1.7.3. □

Corollary 1.7.7. *If G is connected, then $\epsilon \geq \nu - 1$.*

Proof. By Corollary 1.7.6, G contains a spanning tree, say T.
Therefore $\epsilon(G) \geq \epsilon(T)$.
= $\nu(T) - 1,$ by Theorem 1.6.5
= $\nu(G) - 1$
Thus, $\epsilon \geq \nu - 1$. □

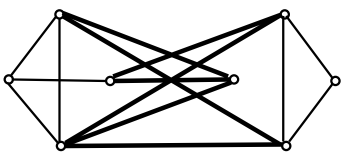
Theorem 1.7.8. *Let T be a spanning tree of a connected graph G and let e be an edge of G not in T. Then $T + e$ contains a unique cycle.*

Proof. Since T is acyclic, each cycle of $T + e$ contains e.
Moreover, C is a cycle of $T + e$ if and only if $C - e$ is a path in T connecting the ends of e.
We know that, in a tree, every pair of vertices are connected by a unique path.
Therefore $T + e$ contains a unique cycle. □

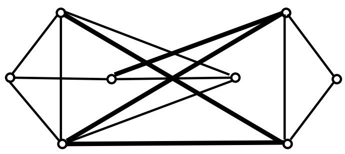
Bonds

Definition 1.7.9. For a subset S ans S' of V, we denote by $[S, S'],$ the set of edges with one end in S and the other end in S'. An *edge cut* of G is a subset of E of the form $[S, \overline{S}]$ where S is a nonempty proper subset of V and $\overline{S} = V - S$.
A minimal nonempty edge cut is called a *bond* of G.

Figure $1.7.3$ gives an edge cut and a bond of a graph.



An edge cut



A bond

Figure $1.7.3$

Remark 1.7.10. Each cut edge $e,$ for instance, gives rise to a bond $\{e\}$.

Remark 1.7.11. If G is connected, then a bond of G is a minimal subset E' of E such that $G - E'$ is disconnected.

Definition 1.7.12. If H is a subgraph of $G,$ the *complement of* H *in* $G,$ denoted by $\overline{H}(G),$ is the subgraph $G - E(H)$. If G is connected, a subgraph of the form $\overline{T},$ where T is a spanning tree, is called a *cotree* of G.

Figure $1.7.4$ represents a spanning tree and its corresponding cotree.

Figure $1.7.4$

Remark 1.7.13. A cotree need not be a tree.

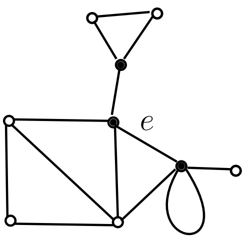
Theorem 1.7.14. *Let T be a spanning tree of a connected graph $G,$ and let e be any edge of $T.$ Then*
(i) the cotree $\overline{T}$ contains no bond of $G;$
(ii) $T+e$ contains a unique bond of $G.$

Proof. (i) Let B be a bond of $G.$
Then $G-B$ is disconnected and so cannot contain the spanning tree $T,$ since T is a connected subgraph of $G.$
Therefore B is not contained in $\overline{T}.$
(ii) Let S denote the vertex set of one of the two components of $T-e.$
The edge cut $B=[S,\overline{S}]$ is clearly a bond of G and is contained in $\overline{T}+e.$
Now, for any $v\in B,$ $T-e+b$ is a spanning tree of $G.$ Therefore every bond of G contained in $\overline{T}+e$ must include every such element $b.$
It follows that B is the only bond of G contained in $\overline{T}+e.$ □

Remark 1.7.15. The relationship between bonds and cotrees is analogous to that between cycles and spanning trees.

Definition 1.7.16. A vertex v of G is a *cut vertex* if $E(G)$ can be partitioned into two nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have just the vertes v in common. If G is loopless and nontrivial, then v is a cut vertex of G if and only if $\omega(G_v)>\omega(G).$

In Figure $1.7.5,$ all the dark vertices are the cut vertices.



Cut vertices of a graph
Figure $1.7.5$

Theorem 1.7.17. *A vertex v of a tree G is a cut vertex of G if and only if $d(v) > 1.$*

Proof. If $d(v) = 0,$ $G \cong K_1$ and, clearly, v is not a cut vertex.
If $d(v) = 1,$ $G - v$ is an acyclic graph with
$nu(G - v) - 1$ edges and thus a tree by Problem 2 in Section 1.6. Hence $\omega(G - v) = 1 = \omega(G)$ and v is not a cut vertex of $G.$

If $d(v) > 1,$ Then there are two vertices u and w adjacent to $v.$ The path uvw is a $(u, w)-$ path in $G.$ Since G is a tree, uvw is the unique $(u, w)-$ path in $G.$
It follows that there is no $(u, w)-$ path in $G - v,$ and hence $G - v$ is disconnected.
Therefore, $\omega(G - v) > \omega(G) = 1.$
Thus v is a cut vertex of $G.$ □

Corollary 1.7.18. *Every nontrivial connected graph without loops has at least two vertices that are not cut vertices.*

Proof. Let G be a nontrivial loopless connected graph.
By Corollary 1.6.6, G contains a spanning tree $T.$ By Corollary 1.7.6 and Therorem 1.7.17, T has at least two vertices that are not cut vertices (by above theorem); let v be one of them. Then $\omega(T - v) = 1.$
Since T is a spanning subgraph of $G,$ $T - v$ is a spanning subgraph of $G - v.$ Thus $\omega(G - v) \leq \omega(T - v).$
It follows that $\omega(G - v) = 1$ and hence that v is not a cut vertex of $G.$ Since There are at least two such vertices $v,$ the proof is complete. □

Solved Problems

Problem 1. A simple cubic connected graph has a cut vertex if and only if it has a cut edge.

Solution. Let G have a cut vertex $v.$
Let $v_1,$ $v_2,$ v_3 be the vertices of G that are adjacent to v in $G.$
Then $G - v$ is disconnected with two or three components. If $G - v$ has three components, no two of $v_1,$ $v_2,$ v_3 can belong to the same component of $G - v.$
In this case, each of the edges vv_1, vv_2 and vv_3 is a cut edge of G (see Figure $1.7.6$)

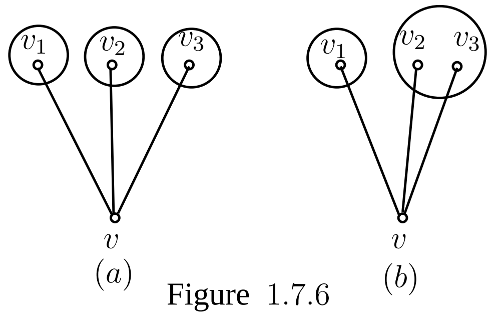


Figure 1.7.6

If $G - v$ has two components, one of the vertices say v_1 belongs to one component of $G - v,$ and the other vertices $v_2,\ v_3$ belong to the other component of $G - v.$ In this case, vv_1 is a cut edge.

Conversely, let $e = uv$ be a cut edge of $G.$

Then $G - uv$ is disconnected with two components, each of which contains at least four vertices, since G is cubic. Therefore, the deletion of u from G disconnects G into two or more components.

Hence u is a cut vertex of $G.$ ■

Problem 2. Prove that G is a forest if and only if every edge of G is a cut edge.

Solution. Assume that G is a forest.

Then each component of G is a tree. Since every edge of a tree is a cut edge, it follows that every edge of G is a cut edge. Conversely, assume that every edge of G is a cut edge.

Suppose that G is not a forest.

Then G has a cycle, say $C.$ By Theorem 1.6.5, every edge on C is not a cut edge of $G,$ which is a contradiction. Hence G is a forest. ■

Problem 3. If a graph with at least three vertices has a cut edge, prove that it has a cut vertex. Is the converse true?

Solution. It suffices to prove the problem for a connected graph.

Let G be a connected graph with at least three vertices.

Let $e = uv$ be a cut edge of $G.$

The $G - e$ is disconnected.

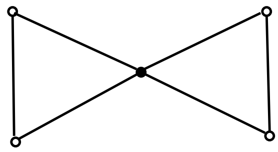
Since the deletion of the vertex u (or v) includes the deletion of the edge $e.$

Since G has at least three vertices, it follows that $G - u$ or $G - v$ is also disconnected.

Hence, u or v is a cut vertex of $G.$

The converse of the problem is not true. That is, a graph with a cut vertex need not have a cut edge.

For example, the dark vertex in the graph in Figure 1.7.7. is a cut vertex, but there is not a cut edge.



A graph with a cut vertex but no cut edge
Figure 1.7.7

Excercises

1. If e is an edge of a connected graph $G,$ prove that e is in every spanning tree of G if and only if e is a cut edge of $G.$

2. If G is a graph without loops but has exactly one spanning tree $T,$ prove that $G = T.$

3. Prove that G has at least $\epsilon - v + \omega$ distinct cycles.

4. If each degree in G is even, prove that G has no cut edge.

5. If G is a k -regular graph with $k > 1,$ prove that G has no cut edge.

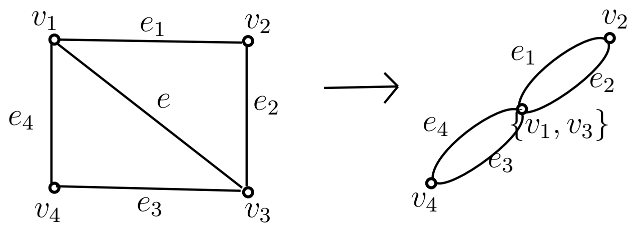
6. If G is a connected graph and S is any nonempty proper subset of $V,$ prove that the edge cut $[S, \overline{S}]$ is a bond of G if and only if both $G[S]$ and $G[\overline{S}]$ are connected.

7. Prove that every edge cut is a disjoint union of bonds.

1.8 Spanning trees

Definition 1.8.1. An edge e of a graph G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G.e$

Figure 1.8.1 illustrates the effects of contracting the edge $e.$

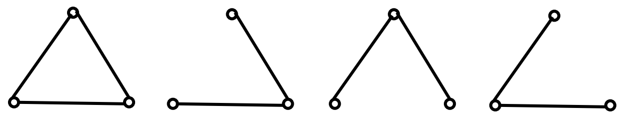


Contraction of an edge
Figure 1.8.1

Remark 1.8.2. It is clear, from its definition, that
$\nu(G.e) = \nu(G) - 1$
$\epsilon(G.e) = \epsilon(G) - 1$ and
$\omega(G.e) = \omega(G)$.
Therefore, if T is a tree, so too is $T.e$.

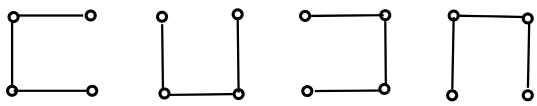
Notation. The number of distinct spanning trees of G is denoted by $\tau(G)$.

Figure 1.8.2. shows all the three distinct spanning trees of C_3.



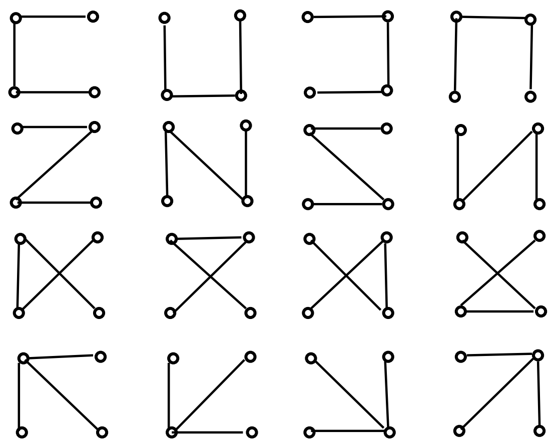
Distinct spanning trees of K_3
Figure 1.8.2

Figure 1.8.3. shows all the four distinct spanning trees of C_4.



Distinct spanning trees of C_4.
Figure 1.8.3

Thus, we see that $\tau(C_3) = 3$ and $\tau(C_4) = 3$. In general, $\tau(C_n) = n$.
The complete graph on four vertices has 16 distinct spanning trees.
They are illustrated in Figure 1.8.4. Thus $\tau(K_4) = 16$.



Sixteen distinct spanning trees of K_4.
Figure 1.8.4

Theorem 1.8.3. ***(Cayley's recursive formula)*** *If e is a link of G, then $\tau(G) = \tau(G - e) + \tau(G.e)$.*

Proof. Since every spanning tree of G that does not contain e is also a spanning tree of $G - e$ and conversely, it follows that $\tau(G_e)$ is the number of spanning trees of G that do not contain e.
Now to each spanning tree T of G that contains e, there corresponds a spanning tree $T.e$ of $G.e$.
This correspondence is clearly a bijection.

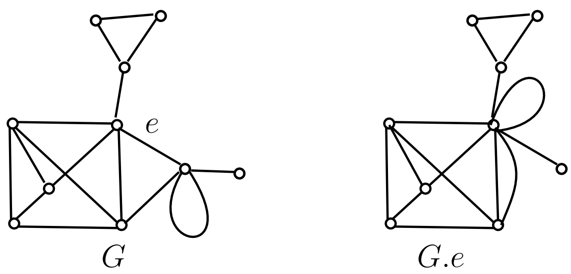


Figure 1.8.5

Therefore, $\tau(G.e)$ is precisely the number of spanning trees of G that contain e.
it follows that $\tau(G) = \tau(G - e) + \tau(G.e)$. $\square$

In the special case when G is complete, a simple formula of $\tau(G)$ was discovered by Cayley in 1889.

Theorem 1.8.4. $(Cayley)$ $\tau(K_n) = n^{n-2}$.

Proof. Let the vertex set of K_n be $N = \{1, 2, \ldots, n\}$.

We note that n^{n-2} is the number of sequences of length $n-2$ that can be formed from N .

Thus to prove the theorem, it suffices to establish a one-one correspondence between the set of spanning trees of K_n and the set of such sequences.

With each spanning tree T of K_n, we associate a unique sequence $\{t_1, t_2, \ldots, t_{n-2}\}$ as follows:

Regarding N as an ordered set, let s be the first vertex of degree one in T"; the vertex adjacent to s is taken as t_1. We now delete s_1 from T, denote by S_2 the first vertex of degree one in $T - s$ and take the vertex adjacent to s_2 as t_2. This operation is repeated until t_{n-2} has been defined and the tree with just two vertices remains. The tree in Figure $1.8.7$, for instance, gives rise to the sequences $(4, 3, 5, 3, 4, 5)$. It can be seten that different spanning trees of K_n determine different sequences.

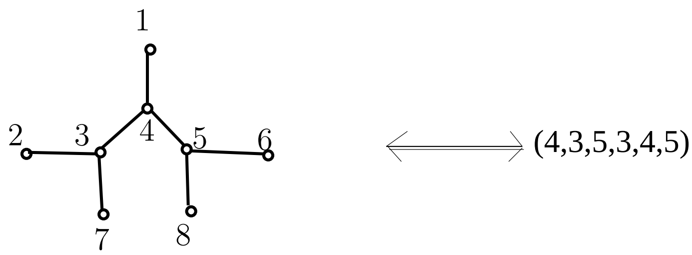


Figure $1.8.7$

The reverse procedure is equally straightforward. Observe, first that any vertex v of T occurs $d_r(v) - 1$ times in $(t_1, t_2, \ldots, t_{n-2})$. Thus the vertices of degree one in T are precisely those that do not appear in this sequence. To reconstruct T from $(t_1, t_2, \ldots, t_{n-2})$ we therefore proceed as follows:

Let s_1 be the first vertex of N not in $(t_1, t_2, \ldots, t_{n-2})$; join s_1 to t_1. Next, let s_2 be the first vertex of $N \backslash \{s_1\}$ not in $(t_1, t_2, \ldots, t_{n-2})$; and join s_2 to t_2. Continue in this way until the $n-2$ edges $s_1t_1, s_2t_2, \ldots, s_{n-2}t_{n-2}$ have been determined. T is now obtained by adding the edge joining the two remaining vertices of $N \backslash \{s_1, s_2, \ldots, s_{n-2}\}$. It is easily verified that different sequences give rise to different spanning trees of K_n. We have thus established the desired one-to-one correspondence. $\square$

Remark 1.8.5. Note that n^{n-2} is not the number of nonisomorphic spanning trees of K_n; but the number of distinct spanning trees of K_n; there are just six nonisomorphic spanning trees of K_6, whereas there are $6^4 = 1296$ distinct spanning trees on K_6.

Definition 1.8.6. A *wheel* is a graph obtained from a cycle by adding a new vertex and joining it with all the vertices of the cycle. The new edges are called the *spokes* of the wheel. A wheel on n vertices is denoted by W_n.

Figure $1.8.8$ shows W_3, W_5 and W_6.

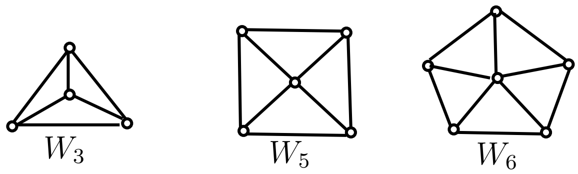


Figure $1.8.8$

Exercises

1. Draw the distinct spanning trees of K_5. How many of them are nonisomorphic?
2. Draw the distinct spanning trees of W_4. How many of them are nonisomorphic?
3. Using Cayley's recursive formula, evaluate the number of spanning trees of $K_{3,3}$.
4. If e is an edge of K_n, prove that $\tau(K_n - e) = (n-2)n^{n-3}$.
5. Obtain an expression for the number of spanning trees of W_n.

Chapter 2

CONNECTIVITY AND EULER TOURS

2.1 Connectivity

Consider the four connected graphs in Figure $2.1.1.$ G_1 is a tree, a minimal connected graph. Deletion of any of the four edges disconnects it. But G_2 cannot be disconnected by the deletion of a single edge, but can be disconnected by the deletion of one vertex, its cut vertex. There are no cut edges or cut vertices in $G_3,$ but even so G_3 is not well connected as $G_4,$ the complete graph on 4 vertices. Thus, intuitively, each successive graph is more strongly connected than the previous one. This leads to the concept of connectivity and edge connectivity which measure the extent to which the graph is connected.

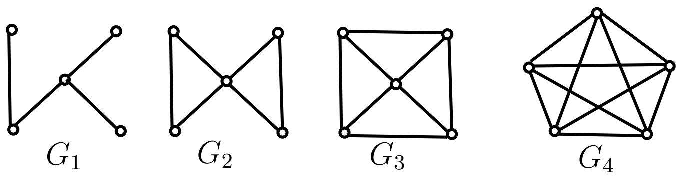


Figure $2.1.1$

Definition 2.1.1. A *vertex cut* of G is a subset V' of V such that $G - V'$ is disconnected. A k *-vertex cut* is a vertex cut of k elements.

A complete graph has no vertex cut; in fact, the only graphs that do not have vertex cuts are those that contain complete graphs as spanning subgraphs.

If G has at least one pair of distinct nonadjacent vertices, the *connectivity* of $G,$

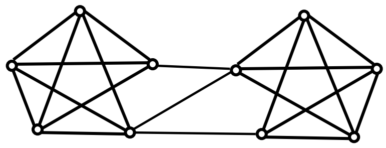
denoted by $\kappa(G),$ is the minimum k for which G has a k -vertex cut; otherwise, we define $\kappa(G)$ to be $\nu - 1.$ Thus, $\kappa(G) = 0$ if G is either trivial or disconnected. The graph G is said to be k *-connected* if $\kappa(G) \geq k.$

All nontrivial connected graphs G are 1 -connected, that is, $\kappa(G) \geq 1.$

Definition 2.1.2. A *edge cut* of G is a subset of $E(G)$ of the form $[S, \overline{S}],$ where S is a nonempty proper subset of $V(G).$ A k *-edge cut* is a edge cut of k elements.

If G is nontrivial and E' is an edge cut of $G,$ then $G - E'$ is disconnected; then we define the *edge connectivity* of $G,$ denoted by $\kappa'(G),$ to be the minimum k for which G has a k -edge cut. If G is trivial, we define $\kappa'(G)$ to be $0.$ Thus, $\kappa'(G) = 0$ if G is either trivial or disconnected, and $\kappa'(G) = 1$ if G is a connected graph with a cut edge. The graph G is said to be k *-edge-connected* if $\kappa'(G) \geq k.$

All nontrivial connected graphs G are 1 -edge-connected, that is, $\kappa'(G) \geq 1.$



Graph with $\kappa = 2,$ $\kappa' = 3$ and $\delta = 4$

Figure $2.1.2$

Theorem 2.1.3. *For any graph* $G,$ $\kappa \leq \kappa' \leq \delta.$

Proof. First we prove the inequality $\kappa' \leq \delta.$ If G is trivial, then $\kappa' = 0 \leq \delta.$ Otherwise, the set of links incident with a vertex of degree δ constitute a δ -edge cut of $G.$ It follows that $\kappa' \leq \delta.$

We now prove that $\kappa \leq \kappa'$ by induction on $\kappa'.$

If $\kappa' = 0,$ then G must be either trivial or disconnected and so $\kappa = 0.$ Therefore, the inequality $\kappa \leq \kappa'$ is true if $\kappa' = 0.$

Assume that the inequality $\kappa \leq \kappa'$ is true for all graphs with edge connectivity less than $k.$

Let G be a graph with $\kappa'(G) = k > 0;$ let e be an edge in a k -edge cut of $G.$

Consider the subgraph $H = G - e.$

Clearly $\kappa'(H) = k - 1$ and so by induction hypothesis, $\kappa(H) \leq \kappa'(H) = k - 1.$

If H contains a complete graph as a spanning subgraph, then so does G and $\kappa(G) = \kappa(H) \leq k - 1.$ Otherwise, let S be a vertex cut of H with $\kappa(H)$ elements. Since $H - S$ is disconnected, one of the following holds:

(i) $G - S$ is disconnected; and

(ii) $G-S$ is connected with a cut edge $e.$

If (i) holds, then $\kappa(G) \le \kappa(H) \le k-1 < k = \kappa'(G).$

If (ii) holds, then either $\nu(G-S) = 2$ or $G-S$ has a 1-vertex cut.

If $\nu(G-S)=2,$ then $\kappa(G) \le \nu(G)-1 = \kappa(H)+1 \le k-1+1 = k = \kappa'(G).$

If $G-S$ has a 1-vertex cut, say $\{v\},$ then $S \cup \{v\}$ is a vertex cut of G and $\kappa(G) \le \kappa(H)+1 \le k = \kappa'(G).$

Thus in each case, we have $\kappa(G) \le \kappa'(G).$

Therefore, the result follows by the principle of induction. □

Remark 2.1.4. The graph in Figure 2.1.2 shows that strict inequality can hold in the above theorem.

Theorem 2.1.5. *A set F of edges in G is an edge cut if and only if F contains an even number of edges from every cycle in $G.$*

Proof. **Necessity.** A cycle must wind up on the same side of an edge cut that it starts on, and thus it must cross the cut an even number of times.

Sufficiency. Given a set F that satisfies the intersection condition with every cycle, we construct a set $S \subset V(G)$ such that $F = [S, \overline{S}].$ Each component of $G-F$ must be all in $G[S]$ or all in $G[\overline{S}],$ but we must group them appropriately. Define a graph H whose vertices correspond to the components of $G-F;$ for each $e \in F,$ we put an edge in H whose endpoints are the components of $G-F$ containing the endpoints of $e.$

We claim that H is bipartite. From a cycle C in $H,$ we can obtain a cycle C' in G as follows. For $v \in V(C)$ let e, f be the edges of C incident to v (not necessarily distinct), and let x, y be the endpoints of e, f in the component of $G-F$ corresponding to $v.$ We expand v into an (x, y) path in that component. Since C visits each vertex at most once, the resulting C' is a cycle in $G.$ The number of edges of F in C' is the length of $C.$ Hence the length of C is even.

We conclude that H is bipartite. Let S be the set of vertices in the components of $G-F$ corresponding to one partite set in a bipartition of $H.$ Now F is the edge cut $[S, \overline{S}].$ □

Lemma 2.1.6. *If G' is obtained from a connected graph G by adding edges joining pairs of vertices whose distance in G is 2, then G' is 2-connected.*

Proof. Since G' is obtained by adding edges to $G,$ G' is also connected. If G' has a cutvertex $v,$ then v is also a cutvertex in $G,$ since $G-v$ is a spanning subgraph of $G'-v.$ By construction, neighbors of v in G are adjacent in $G',$ and hence they cannot be in different components of $G'-v.$ Hence $G'-v$ has only one component. □

Solved Problems

Problem 1. Let G be a 3-connected graph, and let xy be an edge of G. Show that $G-xy$ is 3-connected if and only if $G-\{x,y\}$ is 2-connected.

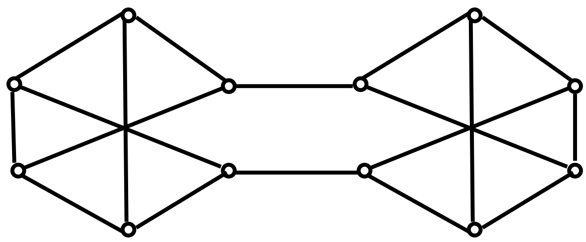
Solution. Given that G is 3-connected with an edge $xy \in E(G)$. Let $G-xy$ is 3-connected. To show $G-\{x,y\}$ is 2-connected. Suppose if possible $G-\{x,y\}$ is not 2-connected. Then there exist a vertex z in $G-\{x,y\}$ which separate $G-\{x,y\}$. Then $\{z, v_{xy}\}$ becomes a separating set of $G-xy$, a contradiction. Hence $G-\{x,y\}$ is 2-connected.

Conversely, suppose $G-\{x,y\}$ is 2-connected. To show $G-xy$ is 3-connected. If possible let $G-xy$ is not 3-connected. Then there exist a separating set $\{u,v\}$ in $G-xy$ which separate $G-xy$. Now if $u,v \neq v_{xy}$ then $\{u,v\}$ becomes a separating set of G, a contradiction. Suppose $u = v_{xy}$. Then v separates $G-\{x,y\}$, a contradiction. Hence $G-xy$ is 3-connected. ■

Problem 2. Every triangle-free simple graph with minimum degree at least 3 and order at most 11 is 3-edge-connected.

Solution. Let $[S,\overline{S}]$ be an edge cut of size less than 3, with $|S| \leq |\overline{S}|$. Let $k=|S|$. Since $\delta(G) \geq 3$ and $[S,\overline{S}] \leq 2$, the fundamental theorem on graphs yields $e(G[S]) \geq \dfrac{(3k-2)}{2}$. Since $G[S]$ is triangle-free, then $e(G[S]) \leq \left\lfloor \dfrac{k^2}{4} \right\rfloor$. Hence $\dfrac{k^2}{4} \geq \dfrac{(3k-2)}{2}$. For positive integer k, this inequality is valid only when $k \geq 6$. Since the smaller side of the cut has at most five vertices, we obtain a contradiction, and there is no edge cut of size at most 2.

The bound of 11 is sharp. The 3-regular triangle-free graph of order 12 shown below is not 3-edge-connected. ■



Problem 3. Show that if G is k -edge connected, then $\epsilon \geq \frac{k\nu}{2}$.
Solution. We know that $\sum\limits_{v \in V(G)} d(v) = 2\epsilon$.
If the vertices are $v_1, v_2, \ldots, v_\nu,$ then $d(v_1) + d(v_2) + \ldots + d(v_\nu) = 2\epsilon$.
Since G is k -edge connected, $\kappa'(G) \geq k$. We know that $\kappa \leq \kappa' \leq \delta$. Hence, $\kappa \leq \delta$.

$$\nu\delta \leq \sum d(v) = 2\epsilon$$

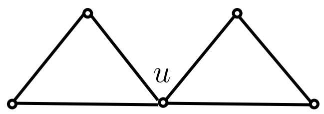
$$\nu k \leq \nu\delta \leq 2\epsilon$$

$$\Rightarrow \nu k \leq 2\epsilon$$

$$\Rightarrow \epsilon \geq \frac{1}{2}\nu k.$$

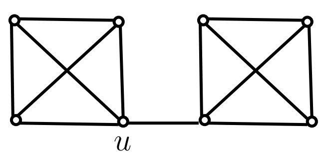
■

Problem 4. Find a simple graph with $\delta = \nu - 3$ and $\kappa < \delta$.
Soluion.



Here $\nu = 5,\ \delta = 2 = \nu - 3.$ Since u is vetex cut, $\kappa = 1 < \delta$. ■

Problem 5. Find a simple graph G with $\delta \geq [\frac{\nu}{2} - 1]$ and $\kappa' < \delta$.
Solution.



Here $\nu = 8,\ \delta = [\frac{\nu}{2} - 1] = [4 - 1] = 3$ and $\kappa' = 1 < \delta(= 3)$. ■

Problem 6. If G is simple and 3 -regular, prove that $\kappa = \kappa^{'}$.

Solution. It suffices to consider only connected cubic graph G. Further since $\kappa \leq \kappa' \leq \delta = 3,$ we have to consider only the cases when $\kappa = 1,\ 2,\ or\ 3.$

By Problem 1 of Section 1.7, we have, for a simple cubic graph, $\kappa = \kappa'$ if $\kappa = 1.$

If $\kappa = 3,$ then by Theorem 2.1.3, $\kappa \leq \kappa' \leq \delta = 3,$ and hence $\kappa' = 3.$

We shall now prove that $\kappa = 2$ implies that $\kappa' = 2.$

Let $\kappa = 2$ and $\{u, v\}$ be a 2 -vertex cut of $G.$ The deletion of $\{u, v\}$ results in a disconnected subgraph G' of $G.$ Since each of u and v must be joined to a vertex of each component of G' and since G is cubic, G' can have at most three components.

If G' has precisely three components G_1, G_2 and $G_3,$ and if e_i and $f_i, i = 1, 2, 3$ join respectively u and v in $G_i,$ then each pair $\{e_i, f_i\}$ is an edge cut of $G.$

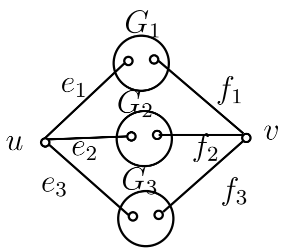


Figure 2.1.3

If G' has only two components G_1 and $G_2,$ then each u and v is joined to one of G_1 and G_2 by a single edge e and f respectively so that $\{e, f\}$ is an edge cut of $G.$

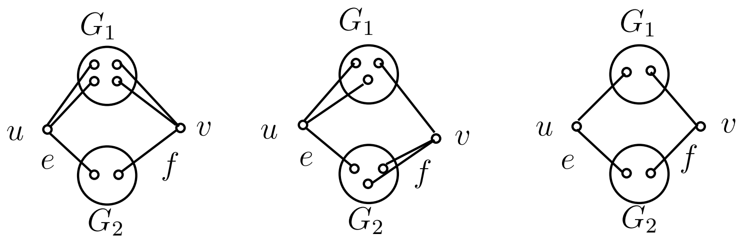


Figure 2.1.4

Thus in either case, there exists an edge cut consisting of two edges. Hence $\kappa' \leq 2.$ But, by Theorem 2.1.3, $\kappa' \geq \kappa = 2.$ Hence $\kappa' = 2.$

Thus $\kappa = \kappa'.$ ∎

Problem 7. Prove that the connectivity of the k -cube is $k.$

Solution. We know that the k -cube Q_k is k -regular. By deleting the neighbours of any vertex we can get the resulting graph disconnected. Hence $\kappa(Q_k) \leq k.$

To prove that $\kappa(Q_k) \geq k,$ we have to prove that any vertex cut of Q_k has at least k vertices. We prove this by induction on $k.$
For $k=1,$ $Q_1 \cong K_2$ and so $\kappa(Q_1)=1.$
Now assume that $k>1$ and $\kappa(Q_{k-1}) \geq k-1.$
Note that Q_k can be obtained from two copies, say Q, Q' of $Q_{k-1},$ by joining the corresponding vertices in Q and $Q'.$ Let S be any arbitrary vertex cut of $Q_k.$

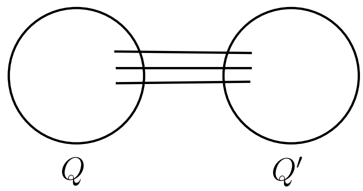


Figure 2.1.4

If both $Q-S$ and $Q'-S$ are connected, then $Q_k - S$ is also connected unless S deletes at least one end vertex of every edge newly added. This requires $|S| \geq 2^{k-1}.$ But $2^{k-1} \geq k$ for $k \geq 2.$ Thus, $\kappa(Q_k) \geq k.$ Now, we may assume that one of them, say $Q-S$ is disconnected. Then S has at least $k-1$ vertices from $V(Q)$ by induction hypothesis. If S contains no vertices of $Q',$ then $Q'-S$ is connected. Since every vertex of $Q-S$ has a neighbour in $Q'-S,$ it follows that $Q-S$ is connected, giving a contradiction. Hence S must also contain a vertex of Q' so that $|S| \geq k.$ Hence $\kappa(Q_k) \geq k.$

Thus, $\kappa(Q_k)=k.$ ■

Problem 8. If G is k-connected, prove that $\epsilon \geq \dfrac{k\nu}{2}.$ Deduce that there is no 3-connected simple graph with 7 edges.
Solution. Since G is k-connected, $\kappa(G) \geq k.$ Further, $\kappa \leq \delta$ by Theorem 2.1.3.
Thus, $\delta \geq \kappa \geq k.$ By the fundamental theorem on graphs, we have
$2\epsilon = \sum d(v) \geq \nu\delta \geq \nu k.$
Therefore, $\epsilon \geq \dfrac{\nu k}{2}.$
If possible, suppose there exists a 3-connected simple graph G with 7 edges.
Since $\epsilon(K_4)=6,$ it follows that $\nu(G) \geq 5.$
Therefore $\epsilon \geq \dfrac{3\nu}{2} = \dfrac{15}{2},$ giving a contradiction. ■

Exercises

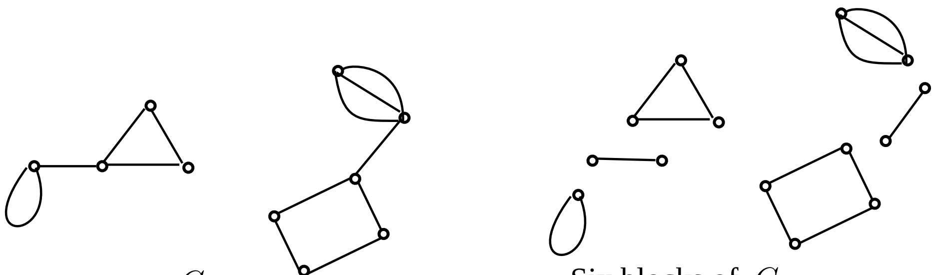
1. Let G be a 2-connected graph with $\delta(G) \geq 3.$ Prove that there exist a vertex $v \in V(G)$ such that $G-v$ is also 2-connected.

2. If G is k-edge connected with $k > 0$ and if E' is a set of k edges of $G,$ prove that $\omega(G - E') \leq 2.$
3. For $k > 0,$ find a k-connected graph G and a set V' of vertices of G such that $\omega(G - V') > 2.$
4. If G is k-edge connected, prove that $\epsilon \leq \frac{k\nu}{2}.$
5. If G is simple and $\delta \leq \nu - 2,$ prove that $\kappa = \delta.$
6. Find a simple graph with $\delta = \nu - 3$ and $\kappa < \delta.$
7. If G is simple and $\delta \geq \nu - 2,$ prove that $\kappa' = \delta.$
8. Find a simple graph G with $\delta \geq [\frac{\nu}{2} - 1]$ and $\kappa' < \delta.$
9. If G is simple and $\delta \geq \frac{\nu + k - 2}{2},$ then prove that G is k-connected.
10. If l, m and n are integers such that $0 < l \leq m \leq n,$ then prove that there exists a simple graph G with $\kappa = l,$ $\kappa' = m$ and $\delta = n.$

2.2 Blocks

Definition 2.2.1. A connected graph with no cut vertices is called a *block*. Every block with at least three vertices is 2-connected. A *block of a graph* is a subgraph that is a block and is maximal with respect to this property. Every graph is the union of its blocks.

Figure $2.2.1$ shows a graph and its blocks.



G Six blocks of G

Figure $2.2.1$

Definition 2.2.2. A family of paths in G is said to be *internally-disjoint* if no vertex of G is an internal vertex of more than one path of the family.

Theorem 2.2.3. *A graph G with $\nu \geq 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally disjoint paths.*

Proof. If any two vertices of G are connected by at least two internally disjoint paths, then clearly G is connected and has no 1-vertex cut. Hence G is 2-connected.

Conversely, let G be a 2-connected graph. We have to prove that any two vertices u and v are connected by at least two internally disjoint paths. We shall prove the result by induction on $d(u, v)$.

Suppose $d(u, v) = 1$. Since G is 2-connected, the edge uv is not a cut edge and therefore it is contained in a cycle. It follows that u and v are connected by two internally disjoint in G.

Now assume that the converse part of the theorem holds for any two vertices at distance less than k and let $d(u, v) = k \geq 2$.

Consider a (u, v)-path of length k and let w be the vertex that precedes v on this path. Since $d(u, w) = k - 1$, it follows from the induction hypothesis that there are two internally disjoint (u, v)-paths, say P and Q in G.
Also since G is 2-connected, $G - w$ is connected and so it contains a (u, v)-path, say P'. Let x be the last vertex of P' that is also in $P \cup Q$. Since u is in $P \cup Q$, there is such an x; we do not exclude the possibility that $x = v$.

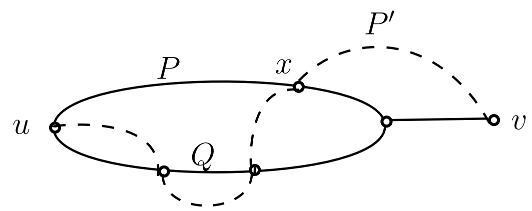


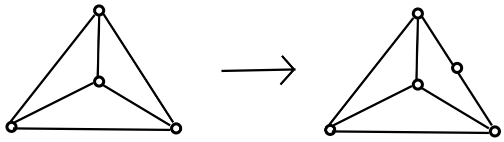
Figure 2.2.2

We may assume, without loss of generality, that x is in P. Then G has two internally disjoint (u, v)-paths, one composed of the $u - x$ section of P together with the $x - v$ section of P', and the other composed of Q together with the path wv. $\square$

Corollary 2.2.4. *If G 2-connected, then any two vertices of G lie on a common cycle.*

Proof. This follows immediately from the above theorem, since two vertices lie on a common cycle if and only if they are connected by two internally disjoint paths. $\square$

Definition 2.2.5. An edge e is said to be subdivided when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex.



Subdivision of an edge
Figure 2.2.3

Above theorem has a generalization to k -connected graphs, known as **Mengers theorem:** a graph G with $\nu \geq k+1$ is k -connected if and only if any two distinct vertices of G are connected by at least k internally disjoint paths.

There is also an edge analogue to this theorem: a graph G is k -edge-connected if and only if any two distinct vertices of G are connected by at least k edge-disjoint paths.

Exercises

1. Prove that a graph is 2 -connected if and only if any two vertices are connected by at least two edge-disjoint paths.
2. Give an example to show that if P is a (u, v) -path in a 2 -connected graph $G,$ then G does not necessarily contain a (u, v) -path internally disjoint from $P.$
3. Let G be a $2-$ connected graph and let X and Y be disjoint subsets of $V,$ each containing at least two vertices. Show that G contains disjoint paths P and Q such that

    (i) the origins of P and Q belong to $X,$

    (ii) the terminus of P and Q belong to $y,$ and

    (iii) no internal vertex of P or Q belongs to $X \cup Y.$
4. Show that a connected graph which is not a block has at least two blocks that contain exactly one cut vertex.
5. Show that the number of blocks in G is equal to $\omega + \sum\limits_{v \in V} (b(v) - 1),$ where $b(v)$ denotes the number of blocks of G containing $v.$
6. Show that if G has no even cycles, then each block of G is either K_2 or an odd cycle.

2.3 Euler Tours

Leonhard Paul Euler (1707- 1783) was a pioneering Swiss mathematician, who spent most of his life in Russia and Germany. Euler (pronounced as OILER) solved the first problem using graph theory and thereby led the foundation of very vast and important field of graph theory. He created first graph to simulate a real time place and situation to solve a problem which was then considered one of the toughest problems.

The Konigsberg Bridge Problem:

The Knigsberg bridge problem originated in the city of Knigsberg, formerly in Germany but, now known as Kaliningrad and part of Russia, located on the river Preger. The city had seven bridges, which connected two islands with the main-land via seven bridges. People staying there always wondered whether was there any way to walk over all the bridges once and only once. The picture shown in Figure 1.1 is the geographic map of Knigsberg during Euler's time showing the actual layout of the seven bridges, highlighting the river Preger and the bridges.

In 1736, Euler came out with the solution in terms of graph theory. He proved that it was not possible to walk through the seven bridges exactly one time. In coming to this conclusion, Euler formulated the problem in terms of graph theory. He drew a picture consisting of dots (vertices) that represented the land masses and the line-segments (edges) representing the bridges that connected those land masses. The resulting picture might have looked somewhat similar to the graph shown in Figure 1.1. This simplifies the problem to great extent. Now, the problem can be merely seen as the way of tracing the graph with a pencil without actually lifting it. One can try it in all possible ways, but you will soon figure out, it is not possible. But Euler not only proved that its not possible, but also explained why it is not and what should be the characteristic of the graphs, so that its edge could be traversed exactly once. He then came out with the new concept of degree of vertices. The degree of a vertex can be defined as the number of edges touching the vertex. Euler proposed that any given graph can be traversed with each edge traversed exactly once if and only if it had, zero or exactly two vertices with odd degrees.

Definition 2.3.1. A trail that traverses every edge of G is called an *euler trail.* A *tour* of G is a closed walk that traverses each edge of G at least once. An euler tour (euler trail) is a tour which traverses each edge exactly once.

An example of a eulerian graph shown below:

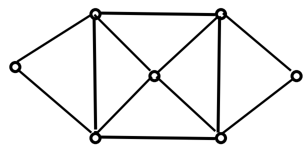


Figure 2.3.3. Euler graph

Theorem 2.3.2. *A nonempty connected graph is eulerian if and only if it has no vertices of odd degree.*

Proof. Let G be eulerian and let C be an euler tour of G with origin (and terminus) $u.$ Each time a vertex v occurs as an internal vertex of $C,$ two of the edges incident with v are accounted for. Since an euler tour contains every edge of $G,$ $d(v)$ is even for all $v \neq u.$ Similarly, since C starts and ends at $u,$ $d(u)$ is also even. Thus G has no vertices of odd degree.

Now, assume, to the contrary, that the sufficiency part does not hold. Then there exists a nonempty noneulerian connected graph with no vertices of odd degree; choose such a graph G with as few edges as possible. Since each vertex of G has degree at least two, G contains a closed trail. Let C be a closed trail of maximum possible length in $G.$ By our assumption, C is not an euler tour of G and so $G - E(C)$ has some component G' with $E(G') > 0.$ Since C itself is eulerian, is has no vertices of odd degree; thus the connected graph G' also has no vertices of odd degree. Since $\epsilon(G') < \epsilon(G),$ it follows from the minimality of $\epsilon(G)$ that G' has an euler tour $C'.$ Now, because G is connected, there is a vertex v in $V(C) \cap V(C')$ and we may assume, without loss of generality, that v is the origin and terminus of both C and $C'.$ But then CC' is a closed trail of G with $\epsilon(CC') > \epsilon(C),$ contradiction to the choice of $C,$ which completes the proof. □

Corollary 2.3.3. *A connected graph has an euler trail if and only if it has at most two vertices of odd degree.*

Proof. If G has an euler trail, then as in the proof of above theorem, each vertex other than the origin and terminus of this trail has even degree in $G.$ Hence G has at most two vertices of odd degree.

Conversely, suppose that G is a nontrivial connected graph with at most two vertices of odd degree. If G has no such vertices, then, by Theorem 2.3.2, G has a closed euler trail. Otherwise, G has exactly two vertices u and v of odd degree. In this case, let $G + e$ denote the graph obtained from G by the addition of a new edge e joining u and $v.$ Clearly, each vertex of $G + e$ has even degree and so by Theorem 2.3.2, $G + e$ has an euler tour $C = v_0 e_1 v_1 \ldots e_{s+1} v_{s+1}$ where $e_1 = e.$ Then the trail $v_1 e_2 v_2 \ldots e_{s+1} v_{s+1}$ is an euler trail of $G.$ □

Solved Problems

Problem 1. If G has no vertices of odd degree, prove that there are edge disjoint cycles $C_1, C_2, \ldots, C_m$ such that $E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_m)$.
Solution. It suffices to prove the problem for connected graphs. Let G be a nontrivial connected graph. Since G has no vertex of odd degree, every vertex is of even degree and hence $\delta(G) \geq 2$. Then we know that G contains a cycle, say C_1. Remove the edges of C_1 from G. We get a spanning subgraph G_1 in which again every vertex has even degree, since only the vertices of C_1 have lost their degree by two in G_1. If G_1 has no edges, then all the edges of G form a cycle and the result is true. Otherwise, G_1 has a cycle, say C_2. As before, remove the edges of C_2 from G_1. We get a spanning subgraph G_2 in which every vertex has even degree. Continuing this process, after some finite number of steps, we get a graph G_m with no edges. Thus, we have got cycles $C_1, C_2, \ldots, C_m$ whose edges form a partition of the edges of G. Thus, $E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_m)$. ■

Problem 2. If a connected graph has $2k \ (> 0)$ vertices of odd degree, prove that there are k edge disjoint trails $Q_1, Q_2, \ldots, Q_k$ such that $E(G) = E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_k)$.
solution Let the $2k$ odd vertices be $v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k$ in any arbitrary order. Construct a new graph G' by adding k edges $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$. Note that G' may be a multi-graph. Now two of these edges are incident at the same vertex. Further, every vertex of G' is of even degree. Hence G' has a closed eulerian trail (euler tour) T. Since no two of these edges are adjacent, it will split into k open trails $Q_1, Q_2, \ldots, Q_k$ whose edges form a partition of the edges of G. Thus, $E(G) = E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_k)$. ■

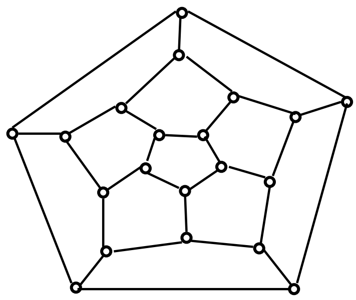
Exercises

1. Does there exist an eulerian graph with
   (a) an even number of vertices and an odd number of edges?
   (b) an odd number of vertices and even number of edges?
   Draw such a graph if exists.
2. Prove that a connected graph G is eulerian if and only if each of its block is eulerian.
3. Prove that a connected graph G is eulerian if and only if each of its edge cut has an even number of edges.

2.4 Hamiltonian Cycles

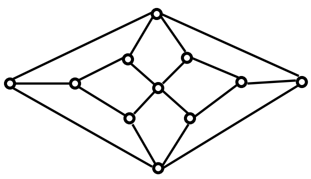
Hamiltonian graphs are named after Sir William Hamilton, an Irish Mathematician (1805-1865), who invented a puzzle, called the Icosian game, which he sold for 25 guineas to a game manufacturer in Dublin. The puzzle involved a dodecahedron on which each of the 20 vertices was labelled by the name of some capital city in the world. The aim of the game was to construct, using the edges of the dodecahedron a closed walk of all the cities which traversed each city exactly once, beginning and ending at the same city. In other words, one had essentially to form a Hamiltonian cycle in the graph corresponding to the dodecahedron. Figure 2.4.1 shows such a cycle.

In contrast with the case of eulerian graphs, no nontrivial necessary and sufficient for a graph to be hamiltonian is known; in fact the problem of finding such a condition is one of the main unsolved problems of graph theory. we will study necessary conditions and sufficient conditions. A multigraph graph is hamiltonian if and only if its underlying graph is hamiltonian, because if G is hamiltonian, then any hamiltonian cycle in G remains a hamiltonian cycle in the underlying graph of G. Conversely, if the underlying graph of a graph G is hamiltonian, then G is also hamiltonian.



The dodecahedron
Figure 2.4.1

Definition 2.4.1. A path that contains every vertex of G is called a *hamiltonian path* of G. A *hamiltonian cycle* of G is a cycle of that contains every vertex of G. A graph is *hamiltonian* if it contains a hamiltonian cycle.



The Harschel graph
Figure 2.4.2

The dodecahedron is hamiltonian and the Herschel graph shown in Figure 2.4.2 is nonhamiltonian.

We shall first present a simple necessary condition.

Theorem 2.4.2. *If G is hamiltonian, then, for every nonempty proper subset S of V, $\omega(G-S) \leq |S|$.*

Proof. Let C be a hamiltonian cycle of G. Then, for every nonempty proper subset S of V, $\omega(C-S) \leq |S|$.
Also, $C-S$ is a spanning subgraph of $G-S$ and so

$$\omega(G-S) \leq \omega(C-S)$$

Therefore, $\omega(G-S) \leq |S|$. $\square$

We now discuss sufficient conditions for a graph G to be hamiltonian. We start with a result due to Dirac.

Theorem 2.4.3. *If G is a simple graph with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$, then G is hamiltonian.*

Proof. We prove by the method of contradiction.
Suppose that the theorem is false. Let G be a maximal nonhamiltonian simple graph with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$.

Since $\nu \geq 3$, G cannot be complete. Let u and v be nonadjacent vertices of G. Since $\delta \geq \frac{\nu}{2}$,

$$d(u) + d(v) \geq \frac{\nu}{2} + \frac{\nu}{2}$$

$$\Rightarrow d(u) + d(v) \geq \nu \qquad \ldots\ldots(1)$$

By the choice of G, $G+uv$ is hamiltonian. Since G is nonhamiltonian, each hamilton cycle of $G+uv$ must contain the edge uv. Thus there is a hamilton path $v_1, v_2, \ldots, v_\nu$ in G with origin $u = v_1$ and terminus $v = v_\nu$. Set

$$S = \{v_i | uv_{i+1} \in E(G)\} \text{ and } T = \{v_i | v_i v \in E(G)\}$$

Since $v_\nu \notin S \cup T$, we have

$$|S \cup T| < \nu \qquad \ldots\ldots(2)$$

We claim that $S \cap T = \phi$

Suppose $S \cap T$ contained some vertex, say v_1, then G would have a hamilton cycle $v_1 v_2 \ldots v_i v_\nu v_{\nu-1} \ldots v_{i+1} v_1$, contradiction to our assumption.

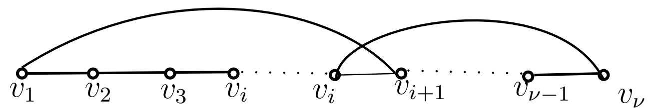


Figure 2.4.3

Hence, $S \cap T = \phi$

$\Rightarrow |S \cap T| = 0 \ldots\ldots (3)$

Using (2) and (3), we obtain

$$\begin{aligned} d(u) + d(v) &= |S| + |T| \\ &= |S \cup T| + |S \cap T| \\ &< \nu \end{aligned}$$

This contradicts (1). Hence the theorem. □

Lemma 2.4.4. *Let G be a simple graph and let u and v be nonadjacent vertices in G such that $d(u)+d(v) \geq \nu$. Then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

Proof. If G is hamiltonian, then trivially $G + uv$ is hamiltonian. Conversely, suppose that $G + uv$ is hamiltonian but G is not hamiltonian. Then as in the proof of the above theorem, we obtain $d(u) + d(v) < \nu$, which contradicts the hypothesis. Therefore, G is hamiltonian. □

Definition 2.4.5. The closure of G is the graph obtained from G by recursively joining the pairs nonadjacent vertices whose degree sum is at least ν until no such pair remains. We denote the closure of G by $c(G)$.

Construction of the closure of a graph on six vertices is shown in Figure 2.4.4.

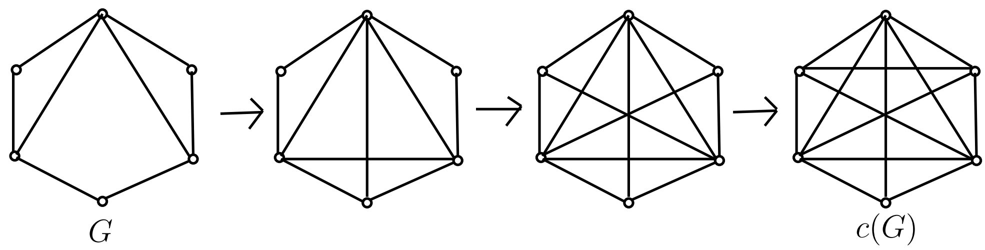


Figure 2.4.4. The closure of a graph

Lemma 2.4.6. $c(G)$ *is well defined.*

Proof. Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains.

Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequence of edges added to G in order to obtain G_1 and G_2 respectively.
We shall show that each e_i is an edge of G_2 as well as each f_i is an edge of G_1.
If possible, let $e_{k+1} = uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_m$ that is not an edge of G_2.
Set $H = G + \{e_1, e_2, \ldots, e_k\}$. It follows form the definition of G_1 that $d_H(u) + d_H(v) \geq \nu$.
By the choice of e_{k+1}, H is a subgraph of G_2.
Therefore, $d_{G_2}(u) + d_{G_2}(v) \geq \nu$.
This is a contradiction, since u and v are nonadjacent in G_2. Therefore, each e_i is an edge of G_2, and similarly, each f_i is an edge of G_1.
Hence, $G_1 = G_2$. Thus $c(G)$ is well defined. □

Theorem 2.4.7. *A simple graph is hamiltonian if and only if its closure is hamiltonian.*

Proof. Apply Lemma 2.4.4 each time an edge is added in the formation of the closure. □

Corollary 2.4.8. *Let G be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then G is hamiltonian.*

Proof. Since complete graphs on at least three vertices are hamiltonian, $c(G)$ is complete and hence it is hamiltonian.
Hence G is hamiltonian by Theorem 2.4.7. □

Remark 2.4.9. Since $c(G)$ is clearly complete when $\delta \geq \frac{\nu}{2}$, Dirac's theorem is an immediate corollary.

Theorem 2.4.10. (***Chvatal***) *Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$ where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of m less than $\dfrac{\nu}{2}$ for which $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Then G is hamiltonian.*

Proof. Let G satisfy the hypothesis. We shall show that its closure $c(G)$ is complete, and the conclusion will then follow from Corollary 2.4.8.
We denote the degree of vertex v in $c(G)$ by $d'(v)$.

Suppose that $c(G)$ is not complete. Let u and v be two nonadjacent vertices in $c(G)$ with

$$d'(u) \leq d'(v) \qquad \ldots(1)$$

and $d'(u) + d'(v)$ is as large as possible.

Since no two nonadjacent vertices in $c(G)$ can have degree sum ν or more, we have
$d'(u) + d'(v) < \nu$ ...(2)
Now denote the set of vertices in $V\backslash\{v\}$ which are nonadjacent to v in $c(G)$ by S and he set of vertices in $V\backslash\{u\}$ which are nonadjacent to u in $c(G)$ by T.
Clearly, $|S| = \nu - 1 - d'(v)$ and
$|V| = \nu - 1 - d(u)$ ...(3)
Furthermore, by the choices of u and v, each vertex in S has degree at most $d'(u)$ and each vertex in $T \cup u$ has degree at most $d'(v)$.

Seting $d'(u) = m$ and using (2) and (3), we find that $c(G)$ has at least m vertices of degree at most m and at least $\nu - m$ vertices of degree less than $\nu - m$. Because G is a spanning subgraph of $c(G)$, the same is also true for G.

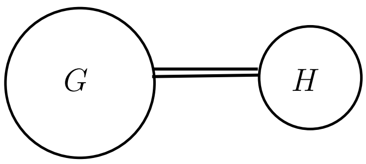
Therefore, $d_m \leq m$ and $d_{\nu-m} < \nu - m$.
But this contradicts the hypothesis, since by (1) and (2) $m < \frac{\nu}{2}$.

Therefore, $c(G)$ is complete.
Hence, G is hamiltonian by Corollary 2.4.8. □

Definition 2.4.11. A sequence of real numbers $(p_1, p_2, \ldots, p_n)$ is said to be *majorised* by another such sequence $(q_1, q_2, \ldots, q_n)$ if $p_i \geq q_i$ for $1 \leq i \leq n$. A graph G is degree majorised by a graph H if $\nu(G) = \nu(H)$ and the nondecreasing degree sequence of G is majorised by that of H.

For instance, the 5-cycle is degree majorised by $K_{2,3}$ because the degree sequence $(2, 2, 2, 2, 2)$ of the 5-cycle is majorised by the degree sequence $(2, 2, 2, 3, 3)$ of $K_{2,3}$.

Definition 2.4.12. Let G and H be two disjoint graphs. Then their *join* $G \vee H$ is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H.



The join of G and H
Figure 2.4.5

Notation
For $1 \le m < \frac{n}{2}$, let $C_{m,n}$ denote the graph $K_m \vee (K_m^c + K_{n-2m})$.
It is illustrated in Figure $2.4.6$.

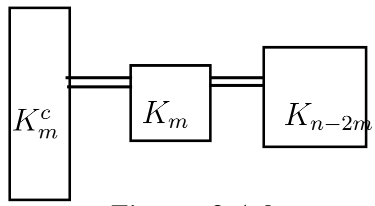


Figure $2.4.6$

$C_{1,5}$ and $C_{2,5}$ are illustrated in Figure $2.4.7$.

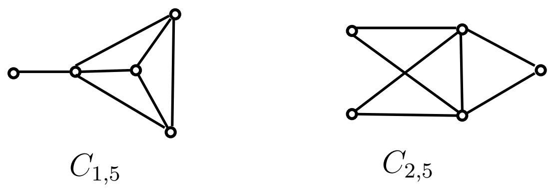


Figure $2.4.7$

Theorem 2.4.13. $C_{m,n}$ *is nonhamiltonian.*

Proof. Let S denote the set of m vertices of degree $n-1$ in $C_{m,n}$.
Then we have $\omega(C_{m,n} - S) = m + 1 > |S|$.
By Theorem 2.4.2, $C_{m,n}$ is nonhamiltonian. $\square$

Theorem 2.4.14. *If* G *is a nonhamiltonian simple graph with* $\nu \ge 3$, *then* G *is degree majorised by some* $C_{m,\nu}$.

Proof. Let G be a nonhamiltonian simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \le d_2 \le \ldots \le d_\nu$ and $\nu \ge 3$. Then by Chvatal Theorem, there exists $m < \frac{\nu}{2}$ such that $d_m \le m$ and $d_{\nu-m} \le \nu - m$. Therefore, $(d_1, d_2, \ldots, d_\nu)$ is majorised by the sequence

$$(m, \ldots, m, \nu - m - 1, \ldots, \nu - m - 1, \nu - 1, \ldots, \nu - 1)$$

with m terms equal to m, $\nu - 2m$ terms equal to $\nu - m - 1$ and m terms equal to $\nu - 1$ and this latter sequence is the degree sequence of $C_{m,\nu}$. $\square$

Corollary 2.4.15. *If* G *is a simple graph with* $\nu \ge 3$ *and* $\epsilon > \binom{\nu-1}{2} + 1$, *then* G *is hamiltonian. Moreover, the only nonhamiltonian simple graph with* ν *vertices and* $\binom{\nu-1}{2} + 1$ *edges are* $C_{1,\nu}$ *and, for* $\nu = 5, C_{2,5}$.

Proof. Let G be a nonhamiltonian simple graph with $\nu \geq 3$. By Theorem 2.4.14, G is degree majorised by $C_{m,\nu}$ for some positive integer $m < \frac{\nu}{2}$. Therefore, by the fundamental theorem on graphs,

$$\begin{aligned}\epsilon(G) &\leq \epsilon(C_{m,\nu}) && \ldots(1)\\ &= \tfrac{1}{2}[m^2 + (\nu - 2m)(\nu - m - 1) + m(\nu - 1)]\\ &= \tbinom{\nu-1}{2} + 1 - \tfrac{1}{2}(m-1)(m-2) - (m-1)(\nu - 2m - 1)\\ &\leq \tbinom{\nu-1}{2} + 1. && \ldots(2)\end{aligned}$$

The degree sequence of $C_{m,\nu}$ is

$$(m, \ldots, m, \nu - m - 1, \ldots, \nu - m - 1, \nu - 1, \ldots, \nu - 1) \qquad \ldots(3)$$

with m terms equal to m, $\nu - 2m$ terms equal to $\nu - m - 1$ and m terms equal to $\nu - 1$. It is clear that the degree sequence (3) is unique, that is any two graphs with degree sequence (3) are isomorphic. Thus, equality can hold only in (1) if G has the same degree sequence as $C_{m,\nu}$ and equality can hold in (2) if either $m = 2$ and $\nu = 5$ or $m = 1$. Hence, $\epsilon(G)$ can equal $\binom{\nu-1}{2} + 1$ only if G has the same degree sequence as $C_{1,\nu}$ or $C_{2,5}$. This implies that $G \cong C_{1,\nu}$ or $G \cong C_{2,5}$. □

Definition 2.4.16. A graph G is *hamilton-connected* if every two vertices of G are connected by a hamilton path. An example of a hamilton-connected graph is as follows:

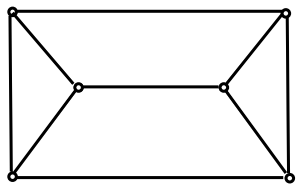


Figure $2.4.8$

Definition 2.4.17. A graph G is hypo-hamiltonian if G is not hamiltonian but $G - v$ is hamiltonian for every $v \in V$. The Petersen graph is hypo-hamiltonian.

Solved Problems

Problem 1. If G is not 2-connected, prove that G is nonhamiltonian.

Solution. If possible, suppose G is hamiltonian, then G contains a spanning cycle C. Hence every pair of vertices of G are connected by two internally disjoint paths along the cycle C. Further, since G is simple, $\nu \geq 3$. Therefore, by Theorem 2.2.3, G is 2-connected, giving a contradiction. ■

Exercise

1. If G is bipartite with bipartition (X, Y), where $|X| \neq |Y|$, prove that G is nonhamiltonian.

2. A mouse eats his way through a 3×3 cube of cheese by tunneling through all of the $271 \times 1 \times 1$ sub cubes. If he starts at one corner and always moves on to an uneaten sub cube, can he finish at the center of the cube?
3. If G has an Hamilton path, then prove that $\omega(G - S) \leq |S| + 1.$
4. Let G be a nontrivial simple graph with degree sequence $(d_1, d_2, \ldots d_\nu)$ where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3.$ If there is no value of m less than $\nu + 1/2$ for which $d_m < m$ and $d_{\nu-m+1} < \nu - m,$ prove that G has a Hamilton path.

Chapter 3

MATCHINGS AND EDGE COLOURINGS

3.1 Matchings

Definition 3.1.1. Le G be a graph. A subset M of E is called a *matching* in G if its elements are links and no two are adjacent in G. The two ends of an edge in M are said to be matched under M. A matching M saturates a vertex v, and v is said to be M *-unsaturated*, if some edge of M is incident with v; otherwise v is said to be M *-unsaturated*.

If each vertex of G is M -saturated, the matching M is called a *perfect matching*.

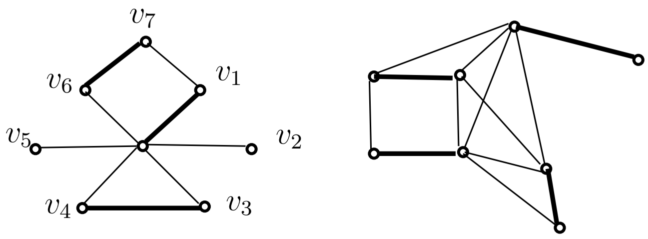
A matching M is called a *maximum matching* if G has no matching M' with $|M'| > |M|$.

Clearly every perfect matching is a maximum matching.

Definition 3.1.2. Let M be a matching in G. An M *-alternating path* in G is a path whose edges are alternatively in $E\backslash M$ an M.
An M *-augmenting path* is an M -alternating path whose origin and terminus are M -unsaturated.

For example, the path $v_5v_8v_1v_7v_6$ in the graph shown in Figure $3.1.1,$ is an M -alternating path.

The set of dark edges of the graphs in Figure $3.1.1$ are a maximum matching and a perfect matching.



A maximum matching A perfect matching

Figure $3.1.1$

Theorem 3.1.3. *A matching* M *in* G *is maximum if and only if* G *contains no* M *-augmenting path.*

Proof. **Necessity.** Let M be a maximum matching in $G.$ We have to prove that G contains no M -augmenting path. Suppose that G contains an M -augmenting path; let it be $v_0v_1v_2v_3\ldots v_{2m+1}.$ Define $M'\subseteq E$ by

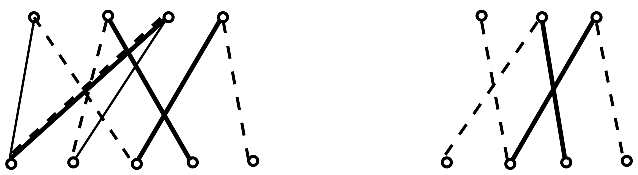
$M'=(M\backslash\{v_1v_2v_3v_4\ldots v_{2m-1}v_{2m}\})\cup\{v_0v_1v_2v_3\ldots v_{2m+1}\}$

Then M' is a matching in G and $|M'|=|M|+1.$ Thus M is not a maximum matching, which is a contradiction. Therefore, G contains no M -augmenting path.

Sufficiency. Assume that G contains no M -augmenting path. We have to prove that M is a maximum matching. Suppose that M is not a maximum matching. Let M' be a maximum matching in $G.$

Then $|M'|>|M|$ . . . (1)

Let $H=G[M\Delta M']$ be the subgraph induced by $M\Delta M',$ where $M\Delta M'$ is the symmetric difference of M and $M'.$



G with M heavy and M' broken $G[M\Delta M']$

Figure $3.1.2$

Each vertex of H has degree either 1 or 2 in $H,$ since it can be incident with at most one edge of M and one edge of $M'.$ Thus each component of H is either an even cycle with edges alternately in M and $M',$ or else a path with edges alternately in M and

$M.$

By (1), H contains more edges of M' than of M and therefore, some path component P of H must start and end with edges of M'. The origin and terminus of $P,$ being M'-saturated in $H,$ are M-unsaturated in $G.$ Thus P is an M-augmenting path in $G,$ which is a contradiction. Therefore, M is a maximum matching. $\square$

Definition 3.1.4. A k*-factor* of G is a k-regular spanning subgraph of $G.$
A graph G is said to be k-factorable if there are edge disjoint k-factors $H_1, H_2, \ldots, H_n$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_n.$ For example, C_4 is 1-factorable and K_5 is 2-factorable as shown in Figure $3.1.3.$

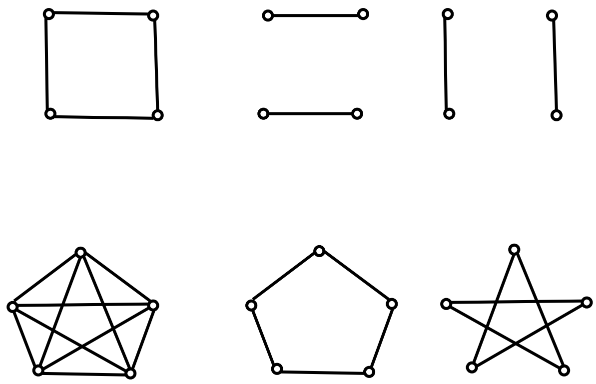


Figure $3.1.3$

Matchings and Coverings in Bipartite Graphs

Assignment Problem

Suppose there are n jobs $j_1, j_2, \ldots, j_n$ in a factory and s workers $w_1, w_2, \ldots, w_s$. Also suppose that each job j_i can be performed by a certain number of workers and that each worker w_i has been trained to do a certain number of jobs. Is it possible to assign each of the n jobs to a worker who can do that job so that no two jobs are assigned to the same worker?

We convert this problem into a graph problem as follows: form a bipartite graph with bipartition (J, W) where $J = \{j_1, j_2, \ldots, j_n\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ and make j_i adjacent to w_k if and only if worker w_k can do the job $j_i.$ Then our assignment problem translate into the following graph problem. Is it possible to find a matching in G that saturates all the vertices of $J?$ A solution to this problem was given by Theorem 3.1.6 due to Hall.

Definition 3.1.5. For any set S of vertices in $G,$ we define the *neighbour set* of S in

G to be the set of all vertices adjacent to vertices in S. This set is denoted by $N_G(S)$.

Theorem 3.1.6. (***Hall***) *Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if*

$|N(S)| \geq |S|$, *for every* $S \subseteq X$.

Proof. Assume that G contains a matching M which saturates every vertex in X. Let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

Conversely, let G be a a bipartite graph with bipartition (X, Y) and $|N(S)| \geq |S|$, for all $S \subseteq X$. Suppose G contains no matching saturating all the vertices in X.

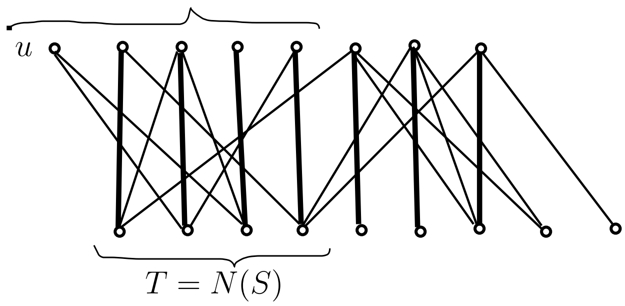


Figure 3.1.4

Let M' be a maximum matching. By our assumption, M' does not saturate all vertices in X. Let u be an M'-unsaturated vertex in X and let Z denote the set of all vertices connected to u by M'-alternating paths. Since M' is a maximum matching, it follows from Theorem 3.1.3 that u is the only M'-unsaturated vertex in Z.

Set $S = Z \cap X$ and $T = Z \cap Y$.

Clearly, the vertices in $S \backslash \{u\}$ are matched under M' with the vertices in T.

Therefore, $|T| = |S| - 1$ ...(1)

$N(S) \subseteq T$.

Since every vertex in $N(S)$ is connected to u by an M'-alternating path,

$N(S) = T$ ...(2)

(1) and (2) imply that

$|N(S)| = |S| - 1 < |S|$, which is a contradiction.

Hence G contains a matching that saturates every vertex in X. $\square$

Corollary 3.1.7. *If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.*

Proof. Let G be a k -regular bipartite graph with bipartition (X, Y).
Since G is k -regular, $k|X| = |E| = k|Y|$,

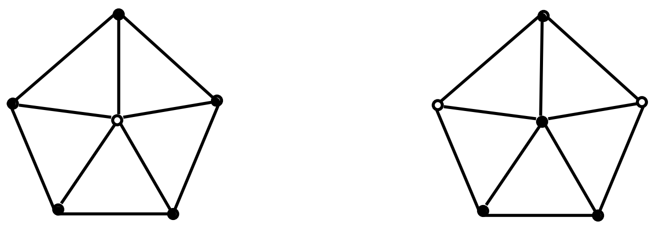
$|X| = |Y|,$ since $k > 0.$

Now let S be a subset of X and denote by E_1 and E_2 the set of edges incident with vertices S and $N(S),$ respectively.
By definition of $N(S), E_1 \subseteq E_2$ and therefore
$k|N(S)| = |E_2| \geq |E_1| = k|S|$
Therefore, $|N(S)| \geq |S|.$

Hence, by Hall's theorem, G has a matching M saturating every vertex in $X.$ Since $|X| = |Y|, M$ is a perfect matching. $\square$

Definition 3.1.8. A *covering* of a graph G is a subset K of V such that every edge of G has at least one in $K.$ A covering K is a *minimimum covering* if G has no covering K' with $|K'| < |K|.$

A covering and a minimum covering of the wheel W_5 are given in Figure 3.1.5.



A covering A minimum covering

Figure $3.1.5$

Remark 3.1.9. If K is a covering of G and M is a matching of $G,$ then K contains at least one end of each of the edges in $M.$ Thus, for any matching M and any covering $K, |M| \leq |K|.$ Indeed, if M' is a maximum matching and K' is a minimum covering, then $|M'| \leq |K'|.$

Lemma 3.1.10. *Let M be a matching and K be a covering such that $|M| = |K|.$ Then M is a maximum matching and K is a minimum covering.*

Proof. If M' is a maximum matching and K' is a minimum covering, then $|M| \leq |M'| \leq |K'| \leq |K|.$

Since $|M| = |K|,$ it follows that

$|M| = |M'|$ and $|K| = |K'|.$ $\square$

Theorem 3.1.11. ($\textbf{\textit{Konig's theorem}}$) *In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.*

Proof. Let G be a bipartite graph with bipartition (X, Y) and let M' be a maximum matching of G.

Denote by U the set of M' -unsaturated vertices in X and denote by Z the set of all vertices connected by M' -alternating paths to vertices of U.
Set $S = Z \cap X$ and $T = Z \cap Y$.

Then as in proof of Hall's theorem, we have that every vertex in T is M' -saturated and $N(S) = T$.

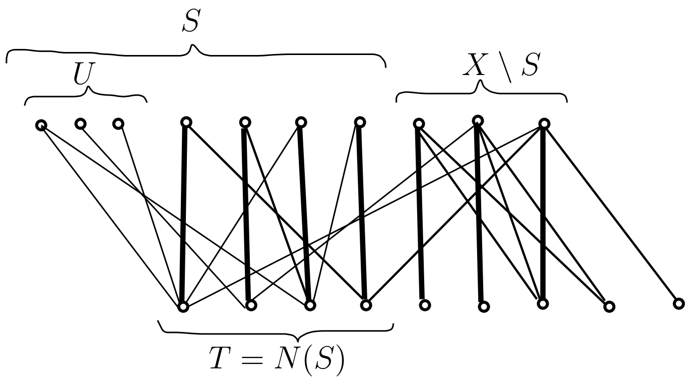


Figure $3.1.6$

Define $K = (X \backslash S) \cup T$.

Every end of G must have at least one of its ends in K. For, otherwise, there would be an edge with one end in S and one end in $Y \setminus T$, contradicting $N(S) = T$.

Thus K is a covering of G and clearly $|M'| = |K|$.
By Lemma 3.1.10, K is a minimum covering and hence the theorem. $\square$

Solved Problems

Problem 1. Find the number of perfect matchings in K_{2n}.

Solution. Let $V(K_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$. The vertex v_1 can be saturated in $2n - 1$ ways by choosing any edge e_1 incident with v_1.
Consider another vertex v_2, saturated by $2n - 3$ ways by choosing any edge e_2 other than e_1, incident with v_2. Continuing this process, the number of perfect matchings in K_{2n} is

$$1.3.5. \cdots .(2n - 1)$$

$$= \frac{1.2.3.\cdots.(2n-1)(2n)}{2.4.6.\cdots.2n}$$
$$= \frac{(2n)!}{2^n.n!}$$

Hence the number of perfect matchings in K_{2n} is $\dfrac{(2n)!}{2^n.n!}$. ■

Problem 2. Prove that a tree G has a perfect matching if and only if $o(G-v) = 1$ for all $v \in V$.

Solution. Assume that G has a perfect matching, say M.
Let u be the vertex matched under M with v. Let G_1 be the component of $G-v$ containing the vertex u. Then, since M is a perfect matching, every other component of $G-v$ are paired under M and so has even order. Since u is paired under M with v, it follows that G_1 has odd order.
Hence $o(G-v) = 1$.
Conversely, assume that $o(G-v) = 1$ for all $v \in V$.
We prove by induction on $\nu(G)$.
If $\nu(G) = 2$, then $G \cong K_2$ and the result is proved.
From our assumption, no two end vertices have the same base. Let v be an end vertex such that its base vertex u has degree two.
Consider $H = G - \{u, v\}$. Then H is a tree of order $\nu(G) - 2$.

Claim: $o(H-w) = 1$ for all $w \in V - \{u, v\}$
Let $w \in V - \{u, v\}$. Let $H_1, H_2, \cdots, H_k$ be the components of $H-w$. Let x be another neighbour of u and $x \in V(H_j)$ for some j. Then in $G-w$, u and v must be confined to a single component namely H_j. Since $o(G-w) = 1$, it follows that the parity in the component of $H-w$ remains unchanged when we add the vertices u and v to $H-w$.
Hence $o(H-w) = 1$ for all $w \in V - \{u, v\}$.
By induction hypothesis, H has a perfect matching, say M^*. Then $M^* \cup \{uv\}$ constitutes a perfect matching in G. ■

Exercises

1. Prove that every k -cube has a perfect matching.
2. Find the number of different perfect matchings in $K_{n,n}$ and K_{2n}.
3. Prove that a tree has at most one perfect matching.

4. For each $k > 1,$ find an example of k -regular simple graph that has no perfect matching.
5. Two people play a game on a graph G alternately selecting different vertices $v_0, v_1, v_2, \ldots$ such that the first player has a winning strategy if and only if G has no perfect matching.
6. Prove that $K_{n,n}$ and K_{2n} are 1 -factorable.
7. Prove that Petersen graph is not 1 -factorable.
8. Prove that a bipartite graph has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V.$
9. For $k > 0,$ prove that every k -regular graph is 1 -factorable.
10. For $k > 0,$ prove that every $2k$ -regular graph is 2 -factorable.

3.2 Tutte's Perfect Matching Theorem

Definition 3.2.1. A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of $G.$ For the graph shown below, $o(G) = 2.$

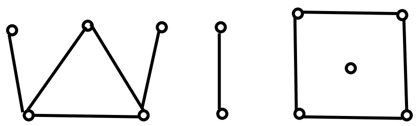


Figure $3.2.1.$ The graph G

Tutte's Perfect Matching Theorem

Theorem 3.2.2. *A graph G has a perfect matching if and only if $o(G-S) \leq |S|,$ for all $S \subsetneq V.$*

Proof. Let G have a perfect matching $M.$ Let S be a proper subset of V and let $G_1, G_2, \ldots, G_n$ be the odd components of $G - S.$

Since G_1 is odd, some vertex u_i of G_i must be matched under M with a vertex v_1 of $S.$ Therefore, since $\{v_1, v_2, \ldots, v_n\} \subset S$

$$o(G-S) = n = |\{v_1, v_2, \ldots, v_n\}| \leq |S|$$

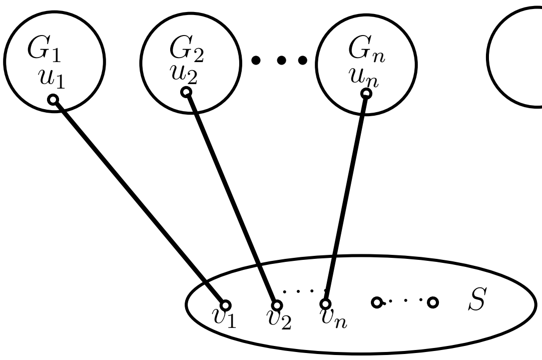


Figure 3.2.2

Conversely, let $o(G-S) \leq |S|$, for all $S \subsetneq V$. We have to prove that G has a perfect matching. Suppose that G has no perfect matching. Then G is a spanning subgraph of a maximal graph G^* having no perfect matching. Since $G-S$ is a spanning subgraph of G^*-S, we have

$$o(G^*-S) \leq o(G-S)$$

and hence by hypothesis,

$$o(G^*-S) \leq |S|, \text{ for all } S \subset V(G^*) \qquad \ldots(1)$$

In particular, setting $S=\phi$, we see that $o(G^*)=0$ and so $\nu(G^*)$ is even.

Denote by U the set of vertices of degree $\nu-1$ in G^*. Since G^* clearly has a perfect matching if $U=V$.
So we assume that $U \neq V$.

We shall show that G^*-U is a disjoint union of complete graphs.

Suppose that some component of G^*-U is not complete. Then, in this component, there are three vetices x, y and z such that $xy \in E(G^*), yz \in E(G^*)$ and $xz \notin E(G^*)$. Since $y \notin U$, there is a vertex w in G^*-u such that $yw \notin E(G^*)$.

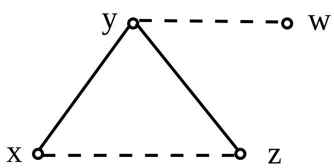
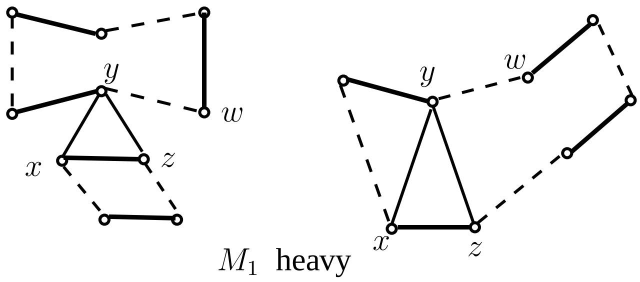


Figure 3.2.3

Since G^* is a maximal graph containing no perfect matching, G^*+e has a perfect matching for all $e \notin E(G^*)$. Let M_1 and M_2 be perfect matchings in G^*+xz and G^*+yw respectively and denote by H the subgraph of $G^* \cup \{xz, yw\}$ induced by

$M_1 \Delta M_2$. Since each vertex of H has degree two, H is a disjoint union of cycles. Also all of these cycles are even, since the edges of M_1 alternate with edges of M_2. We distinguish two cases:



M_1 heavy
(a) M_2 broken (b)
Figure 3.2.4

Case 1. xz and yw are in different components of H (Figure 3.2.4(a)) Then, if yw is in the cycle C of H, the edges of M_1 in C, together with the edges of M_2 not in C, constitute a perfect matching in G^*, contradicting the definition of G^*.
Case 2. xz and yw are in the same components of H. By symmetry of x and z, we may assume that the vertices x, y, w and z occur in that order on C (Figure 3.2.4(b)). Then the edges of M_1 in the section $yw \ldots z$ of C, together with the edge yz and the edges of M_2 not in the section $yw \ldots z$ of C, constitute a perfect matching in G^*, contradicting the definition of G^*.

Since both the cases lead to contradictions, it follows that $G^* - U$ is indeed a disjoint union of complete graphs.

Now, by (1), $o(G^* - U) \leq |U|$. Thus at most $|U|$ of the components of $G^* - u$ are odd. But G^* clearly has a perfect matching: one vertex in each odd component of $G^* - U$ is matched with a vertex of U; the remaining vertices in U and in components of $G^* - u$, are then matched as shown in Figure $3.2.5$.

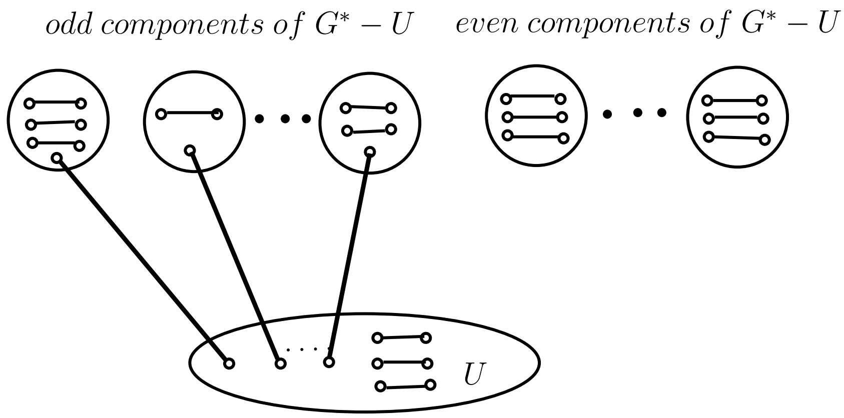


Figure $3.2.5$

Since G^* was assumed to have no perfect matching, we have obtained the desired contradiction.
Thus G does indeed have a perfect matching. $\square$

Corollary 3.2.3. *Every 3-regular graph without cut edges has a perfect matching.*

Proof. Let G be a 3-regular graph without cut edges and let S be a proper subset of $V.$ Denote by $G_1, G_2, \ldots G_n,$ the odd components of $G - S$ and let m_i be the number of edges with one end in G_i and one end $S, 1 \leq i \leq n.$ Since G is 3-regular,

$$\sum_{v \in v(G_i)} d(v) = 3\nu(G_i) \text{ for } 1 \leq i \leq n \ldots\ldots (1)$$

and $\sum_{v \in S} d(v) = 3|S| \ldots\ldots (2)$

By (1), $m_i = \sum_{v \in v(G_i)} d(v) - 2\epsilon(G_i)$ is odd. Now, $m_i \neq 1,$ since G has no cut edges.

Therefore, $m_i \geq 3,$ for $1 \leq i \leq n \ldots\ldots (3)$

It follows from (2) and (3) that

$$\omega(G - s) = n \leq \frac{1}{3} \sum_{i=1}^{n} m_i$$

$$\leq \frac{1}{3} \sum_{v \in S} d(v) = |S|$$

Hence by Tutte's theorem, G has a perfect matching.

$\square$

Remark 3.2.4. A 3-regular graph with cut edges need not have a perfect matching. For instance, the graph shown in Figure 3.2.6 has no perfect matching by Tutte's theorem, since $o(G - v) = 3$.

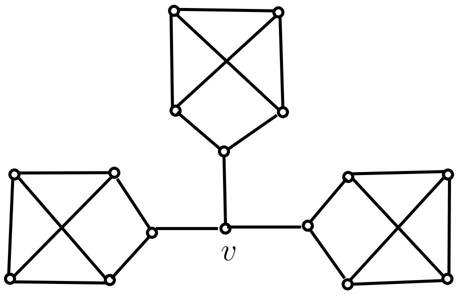


Figure $3.2.6$

Exercises

1. Prove that a tree has a perfect matching if and only if $o(G - v) = 1$ for all $v \in V$.

3.3 Edge Colouring

Definition 3.3.1. Let G be a loopless graph. A k *-edge colouring* ℓ of G is an assignment of k colours, say $1, 2, \ldots, k$ to the edges of G. The colouring ℓ is proper if no two adjacent edges have the same colour.

A k *-edge colouring* can be thought of as a partition $(E_1, E_2, \ldots, E_k)$ of E, where E_i denotes the (possibly empty) subset of edges that have colour i. A *proper* k *-edge colouring* is then a k -edge colouring $(E_1, E_2, \ldots, E_k)$ in which each subset E_i is a matching.

The graph given below has the proper 4 -edge colouring $(\{a, g\}, \{b, e\}, \{c, f\}, \{d\})$.

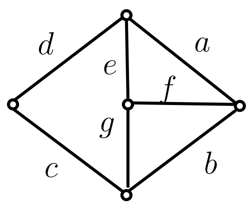


Figure $3.3.1$

Definition 3.3.2. A graph G is said to be k *-edge-colourable* if it has a proper k -edge colouring. Clearly, every loopless graph is ϵ -edge-colourable and if G is k -edge-colourable, then G is also l -edge colourable for every $l > k.$

Definition 3.3.3. The *edge chromatic number* $\chi'(G),$ of a loopless graph is the minimum number k for which G is k -edge-colourable.
G is k -edge-chromatic if $\chi'(G) = k.$

Note that the graph in Figure $3.3.1$ has no proper 3 -edge colouring and hence the graph is 4 -edge-chromatic.

Clearly, in any proper colouring, the edges incident with any one vertex must be assigned different colours. Therefore,

$$\chi' \geq \Delta. \qquad \ldots(1)$$

In the graph shown in Figure $3.3.1$, we have $\chi'(G) = 4$ and $\Delta(G) = 3.$ Hence we observe that the inequality in (1) may be strict. But in case of bipartite graphs, $\chi' = \Delta.$
We say that colour i is said to be *represented at vertex* v if some edge incident with v has colour $i.$

Lemma 3.3.4. *Let* G *be a connected graph that is not an odd cycle. Then* G *has a* 2 *-edge colouring in which both colours are represented at each vertex of degree at least two.*

Proof. We may clearly assume that G is nontrivial.
Case 1. G is eulerian.

If G is an even cycle, the proper 2 -edge colouring of G has the required property. Otherwise, G has a vertex v_0 of degree at least four. Let $v_0e_1v_1e_2\ldots e_\epsilon v_0$ be an euler tour of $G,$ and set

$$E_1 = \{e_i |\ i \text{ is odd}\} \text{ and } E_2 = \{e_i |\ i \text{ is even}\} \qquad \ldots(1)$$

Then the 2-edge colouring (E_1, E_2) of G has the required property, since each vertex of G is an internal vertex of $v_0e_1v_1e_2\ldots e_\epsilon v_0.$
Case 2. G is not eulerian.

In this case, we construct a new graph G^* by adding a new vertex v_0 and joining it to each vertex of odd degree in $G.$ Clearly G^* is eulerian. Let $v_0e_1v_1e_2\ldots e_\epsilon v_0$ be an euler tour of G^* and set E_1 and E_2 as in (1).
It can be easily verified that $(E_1 \cap E, E_2 \cap E)$ has the required property. □

Given a k -edge colouring ℓ of $G,$ we denote by $c(v)$ the number of distinct colours represented at $v.$ Clearly,

$$c(v) \leq d(v). \qquad \ldots(2)$$

Moreover, ℓ is a proper k-edge colouring if and only if equality holds in (2) for all vertices ν of $G.$

Definition 3.3.5. A k-edge colouring ℓ' is an improvement on ℓ if

$$\sum_{v \in V} c'(v) > \sum_{v \in V} c(v), \qquad \ldots(3)$$

where $c'(v)$ is the number of distinct colours represented at v in the colouring $\ell'.$

Definition 3.3.6. An *optimal* k *-edge colouring* is one which cannot be improved.

Lemma 3.3.7. *Let* $\ell = (E_1, E_2, \ldots, E_k)$ *be an optimal* k *-edge colouring. If there is a vertex* u *in* G *and colours* i *and* j *such that* i *is not represented at* u *and* j *is not represented at least twice at* $u,$ *then the component of* $G[E_i \cup E_j]$ *that contains* u *is an odd cycle.*

Proof. Let u be a vertex that satisfies the hypothesis of the lemma, and H denote the component of $G[E_i \cup E_j]$ containing $u.$ Suppose that H is not an odd cycle. Then, by Lemma 3.3.4, H has a 2-edge colouring in which both the colours are represented at each vertex of degree at least two in $H.$ When we recolour the edges of H with colours i and j in this way, we obtain a new edge colouring $\ell' = (E'_1, E'_2, \ldots, E'_k)$ of $G.$ Let $c'(v)$ denote the number of distinct colours at v in the colouring $\ell'.$ We have

$$c'(u) = c(u) + 1$$

since, now, both i and j are represented at $u,$ and also

$$c'(v) \geq c(v), \text{ for } v \neq u.$$

Therefore, $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v),$ contradicting the choice of $\ell.$

Hence, H is indeed an odd cycle. $\square$

Theorem 3.3.8. *If* G *is bipartite, then* $\chi' = \Delta.$

Proof. Since, $\chi' \geq \Delta,$ it is enough to prove that χ' is not strictly greater than $\Delta.$ Suppose $\chi' > \Delta.$ Let $\ell = (E_1, E_2, \ldots, E_\Delta)$ be an optimal Δ-edge colouring of $G,$ and let u be a vertex such that $c(u) < d(u).$ Clearly, u satisfies the hypothesis of Lemma 3.3.7. Therefore, G contains an odd cycle and so is not bipartite, which is a contradiction. Hence, $\chi' = \Delta.$ $\square$

Theorem 3.3.9. (***Vizing's Theorem***) *If* G *is simple, then either* $\chi' = \Delta$ *or* $\chi' = \Delta + 1.$

Proof. Let G be a simple graph. Since, $\chi' \geq \Delta,$ it is enough to prove that $\chi' \leq \Delta + 1.$

Suppose $\chi' > \Delta + 1.$

Let $\ell = (E_1, E_2, \ldots, E_{\Delta+1})$ be an optimal $(\Delta + 1)$ -edge colouring of $G,$ and let u be a vertex such that $c(u) < d(u).$ Then there exist colours i_0 and i_1 such that i_0 is not represented at u and i_1 is represented at least twice at $u.$ Let uv_1 have colour i_1 as in Figure 3.3.2.

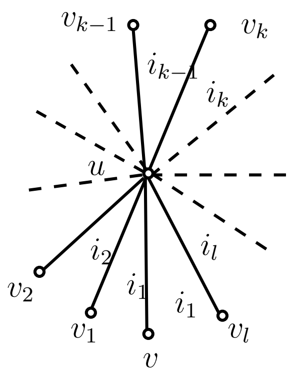


Figure 3.3.2

Since, $d(v_1) < \Delta + 1,$ some colour, say i_2 is not represented at $v_1.$ Now i_2 must be represented at u since otherwise, by recolouring uv_1 with $i_2,$ we would obtain an improvement of $\ell.$ Thus, some edge uv_2 has colour $i_2.$ Again, since $d(v_2) < \Delta + 1,$ some colour i_3 is not represented at v_2 and i_3 must be represented at u since otherwise, by recolouring uv_1 with $i_2,$ and uv_2 with $i_3,$ we would obtain an improved $(\Delta + 1)$ -edge colouring. Continuing this procedure, we construct a sequence $v_1, v_2, \ldots$ of vertices and a sequence $i_1, i_2, \ldots$ of colours such that

(i) uv_j has colour i_j and

(ii) i_{j+1} is not represented at $v_j.$

Since the degree of u is finite, there exists a smallest integer l such that, for some $k < l,$

$(iii) i_{l+1} = i_k.$

The situation is depicted in Figure 3.3.2.

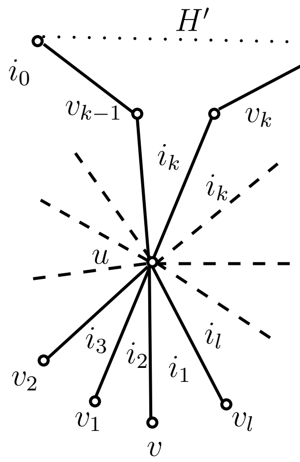


Figure 3.3.3

We now recolour G as follows. For $1 \leq j \leq k-1$, recolour uv_j with colour i_{j+1}, yielding a new $(\Delta+1)$ -edge colouring $\ell' = (E'_1, E'_2, \ldots, E'_{\Delta+1})$ (Figure 3.3.3).
Clearly,

$$c'(v) \geq c(v) \text{ for all } v \in V.$$

Hence, ℓ' is also an optimal $(\Delta+1)$ -edge colouring of G. By Lemma 3.3.7, the component H' of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u is an odd cycle.
Now, in addition, recolour uv_j with colour i_{j+1}, for $k \leq j \leq l-1$ and uv_l with colour i_k, to obtain a $(\Delta+1)$ -edge colouring $\ell'' = (E''_1, E''_2, \ldots, E''_{\Delta+1})$ (see Figure 3.3.4).

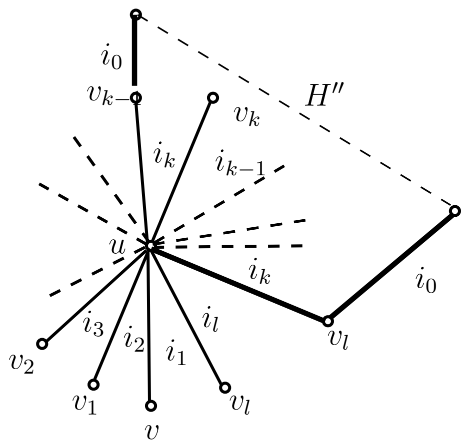


Figure 3.3.4

As above $c''(v) \geq c(v)$ for all $v \in V$ and the component H'' of $G[E''_{i_0} \cup E''_{i_k}]$ that contains u is an odd cycle. But since v_k has degree two in H', v_k clearly has degree one in H''. This contradiction establishes the theorem. $\square$

Remark 3.3.10. Vizing proved a more general theorem than that given above, one that is valid for all graphs without loops. The maximum number of edges joining two vertices in G is called the multiplicity of G denoted by $\mu(g)$.

Vizing's General Theorem: If G is a loopless graph, then $\Delta \leq \chi' \leq \Delta + \mu$.
This theorem is best possible in the sense that, for any $\mu,$ there exists a graph G such that $\chi' = \Delta + \mu.$ For example, in the graph G given in Figure $3.3.5,$ $\Delta = 2\mu$ and since any two edges are adjacent, $\chi' = \epsilon = 3\mu.$

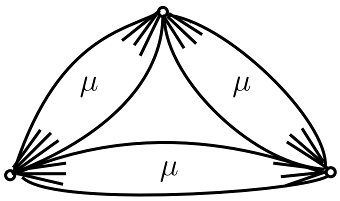


Figure $3.3.5$

Definition 3.3.11. A graph G is called *uniquely* k *-edge colourable* if any two proper k -edge colourings of G induce the same partition on $E.$ For example, the graph given below is uniquely 2 -edge colourable.

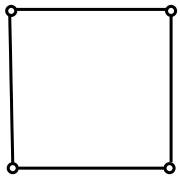


Figure $3.3.6$

Exercises

1. If G is a nonempty regular simple graph with odd number of vertices, prove that $\chi' = \Delta + 1.$
2. If G is a simple graph with $\nu = 2n + 1$ and $\epsilon > n\Delta,$ prove that $\chi' = \Delta + 1.$
3. If G is obtained from a simple regular graph with even number of vertices by subdividing one edge, prove that $\chi' = \Delta + 1.$
4. If G is obtained from a simple regular graph with odd number of vertices by deleting fewer than $\frac{k}{2}$ edges, prove that $\chi' = \Delta + 1.$
5. Prove that every uniquely 3 -edge colourable 3 -regular graph is hamiltonian.

Chapter 4

INDEPENDENT SETS AND CLIQUES

4.1 Independent Sets

Definition 4.1.1. A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set is *maximum* if G has no independent set S' with $|S'| > |S|$.

In the graph given in Figure 4.1.1, $\{u, x\}$ is an independent set of G, while $\{u, w, y\}$ is a maximum independent set of G.

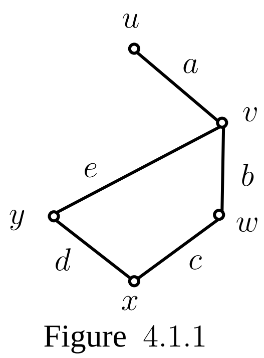


Figure 4.1.1

Definition 4.1.2. The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. Similarly, the number of vertices in a minimum covering of G is called the *covering number* of G and is denoted by $\beta(G)$.

Theorem 4.1.3. *A set $S \subset V$ is an independent set of G if and only if $V \backslash S$ is a* *covering of* G.

Proof. The set S is an independent set of G

$\Leftrightarrow$ no edge of G has both ends in S

$\Leftrightarrow$ each edge has at least one end in $V\backslash S$

$\Leftrightarrow$ $V\backslash S$ is a covering of G $\square$

Corollary 4.1.4. $\alpha + \beta = \nu$.

Proof. Let S be a maximum independent set of G and let K be a minimum covering of G. Then, by Theorem 4.1.3, $V\backslash K$ is an independent set and $V\backslash S$ is a covering. Therefore,

$$\nu - \beta = |V\backslash K| \leq \alpha \qquad \ldots(1)$$

and $$\nu - \alpha = |V\backslash S| \leq \beta \qquad \ldots(2)$$

From (1) and (2), $\alpha + \beta = \nu$. $\square$

Definition 4.1.5. The *edge analogue* of an independent set is a set of links no two of which are adjacent, that is a matching. The edge analogue of a covering is called an *edge covering*. An *edge covering* of G is a subset L of E such that each vertex of G is an end of some edge in L.

Matchings and edge coverings are related to one another as simply as are independent sets and coverings; the complement of a matching need not be an edge covering, nor is the complement of an edge covering necessarily a matching. However, it so happens that the parameters α' and β' are related precisely the same manner as are α and β.

Theorem 4.1.6. (***Gallai***) *If* $\delta > 0$, *then* $\alpha' + \beta' = \nu$.

Proof. Let M be a maximum matching in G and let U be the set of M-unsaturated vertices. Since $\delta > 0$ and M is maximum, there exists a set E' of $|U|$ edges, one incident with each vertex in U. Clearly, $M \cup E'$ is an edge covering of G, and so

$$\begin{aligned}\beta' &\leq |M \cup E'| \\ &= \alpha' + (\nu - 2\alpha') \\ &= \nu - \alpha'\end{aligned}$$

$$\alpha' + \beta' \leq \nu \ldots\ldots (1)$$

Now, let L be a minimum edge covering of G, set $H = G[L]$ and let M be a maximum matching in H. Let U denote the set of M-unsaturated vertices in H. Since M

is maximum, $H[U]$ has no links and therefore,

$$\begin{aligned} |L| - |M| &= |L\ M| \\ &\geq |U| \\ &= \nu - 2|M| \end{aligned}$$

Because, H is a subgraph of G, M is a matching in G and so

$$\alpha' + \beta' \geq |M| + |L| \geq \nu \qquad \ldots(2)$$

From (1) and (2), $\alpha' + \beta' = \nu$. □

Theorem 4.1.7. *In a bipartite graph G with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.*

Proof. Let G be a bipartite graph with $\delta > 0$.
Then $\alpha + \beta = \alpha' + \beta'$ by Theorem 4.1.6 and Corollary 4.1.4
Since G is bipartite, it follows from Theorem 3.1.11, $\alpha' = \beta$.
Therefore, $\alpha = \beta'$. □

Exercises

1. Show that G is bipartite if and only if $\alpha(H) \geq \frac{v(H)}{2}$ for every subgraph H of G.
2. Show that G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G such that $\delta(H) > 0$.
3. A graph is α -critical if $\alpha(G - e) > \alpha(G)$ for all $e \in E$. Show that a connected α -critical graph has no cut vertices.
4. A graph is β -critical if $\beta(G - e) < \beta(G)$ for all $e \in E$. Show that a connected β -critical graph has no cut vertices.

4.2 Ramsey Number

Definition 4.2.1. A *clique* of a simple graph G is a subset S of V such that $G[S]$ is complete.

For the graph G in Figure 4.1.3, $S_1 = \{a, b, c\}$ is a clique since it induces K_3, but $S_2 = \{a, b, c, d\}$ is not a clique since its induced subgraph is not K_4. Some other cliques are $S_3 = \{a, b, c, e\}$ and $S_4 = \{a, e, f\}$.

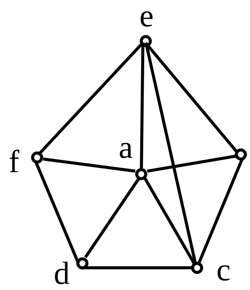


Figure $4.1.3$ The graph G

Remark 4.2.2. Clearly, S is a clique of G if and only if S is an independent set of G^c and so the two concepts are complementary.

Definition 4.2.3. Given any positive integers k and $l,$ there exists a smallest integer $r(k,l)$ such that every graph on $r(k,l)$ vertices contain either a clique of k vertices or an independent set of l vertices. The number $r(k,l)$ is known as the *Ramsey number*.

Remark 4.2.4. $r(1,l) = r(k,1) = 1$ and $r(2,k) = r(k,2) = k$

Theorem 4.2.5. *For any two integers* $k \geq 2$ *and* $l \geq 2,$
$r(k,l) \leq r(k,l-1) + r(k-1,l).$
Furthermore, if both $r(k,l-1)$ *and* $r(k-1,l)$ *are both even, then strict inequality holds.*

Proof. Let G be a graph on $r(k,l-1)+r(k-1,l)$ vertices and let $v \in V.$
We distinguish two cases:

(i) v is nonadjacent to a set S of at least $r(k,l-1)$ vertices, or

(ii) v is adjacent to a set T of at least $r(k-1,l)$ vertices.

Clearly either case (i) or case (ii) must hold because the number of vertices nonadjacent to v together with the number of vertices adjacent to v is equal to $r(k,l-1)+r(k-1,l)+1.$

In case (i), $G[S]$ contains either a clique on k vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Similarly, in case (ii), $G[T \cup \{v\}]$ contains either a clique of k vertices or an independent set of l verrtices. Since one of the cases (i)and (ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices.

Hence, $r(k,l) \leq r(k,l-1) + r(k-1,l).$
Now suppose that $r(k,l-1)$ and $r(k-1,l)$ are both even, and let G be a graph on $r(k,l-1)+r(k-1,l)-1$ vertices. Then the order of G is odd and so not all vertices

have odd degree. Hence some vertex $v \in G$ is of even degree. In particular, v cannot be adjacent to precisely $r(k-1,l)-1$ vertices. Consequently, either case (i) or case (ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices.

Thus, $r(k,l) \le r(k,l-1) + r(k-1,l) - 1.$ □

In general, the determination of Ramsey number is a very difficult unsolved problem. Lower bounds can be obtained from the construction of suitable graphs.

Problem 1. $r(k,l) = r(l,k).$

Solution. Let $r(k,l) = m.$ Let G be any graph on m vertices. Then G^c also has m vertices. Since $r(k,l) = m,$ G^c has either K_k or $\overline{K_l}$ as an induced subgraph. Hence G has either K_l or $\overline{K_k}$ as an induced subgraph. Thus every graph on m vertices contains K_l or $\overline{K_k}$ as an induced subgraph.

Hence, $r(l,k) \le m.$

i.e $r(l,k) \le r(k,l)$ $\ldots(1)$

Interchanging l and $k,$ we get

$r(k,l) \le r(l,k)$ $\ldots(2)$

From (1) and (2), we get $r(k,l) = r(l,k).$ ■

Problem 2. $r(3,3) = 6.$

Solution. From Theorem 4.2.5,

$r(3,3) \le r(3,2) + r(2,3)$

$= 3 + 3$ by remark.

$= 6.$

Therefore, we get $r(3,3) \le 6$ $\ldots(1)$

But the cycle C_5 contains no clique of 3 vertices and no independent set of 3 vertices.

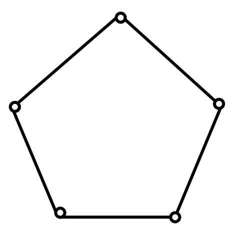


Figure 4.2.1

Thus, $r(3,3) \ge 6$ $\ldots(2)$

From (1) and (2), $r(3,3) = 6.$ ■

Problem 3. $r(3,4) = 9.$

Solution. From Theorem 4.2.5,

$$\begin{aligned} r(3,4) &< r(3,3) + r(2,4) \\ &= 6 + r(2,4), \text{ by Problem 2.} \\ &= 6 + 4, \text{ by the above remark.} \\ &= 10 \end{aligned}$$

Thus, $r(3,4) < 10$ i.e $r(3,4) \leq 9$ $\quad\ldots(1)$

But the graph on 8 vertices shown in Figure $4.2.2$ has no clique of 3 vertices and no independent set of 4 vertices.

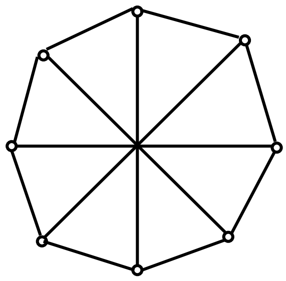


Figure $4.2.2$

Thus, $r(3,4) \geq 9$ $\quad\ldots(2)$

From (1) and (2), $r(3,4) = 9.$ ∎

Problem 4. $r(3,5) = 14.$

Solution. From Theorem 4.2.5,

$$\begin{aligned} r(3,5) &< r(3,4) + r(2,5) \\ &= 9 + 5, \text{ by the above remark} \\ &= 14. \text{ Thus, } r(3,5) < 14 \end{aligned}$$ $\quad\ldots(1)$

But the graph on 13 vertices shown in Figure $4.2.3$ has no clique of 3 vertices and no independent set of 5 vertices.

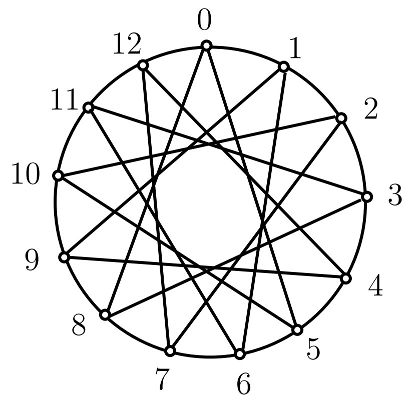


Figure 4.2.3

Thus, $r(3,5) \geq 14$ ...(2)

From (1) and (2), $r(3,5) = 14$.

Problem 5. r(4,4)=18.

Solution. From Theorem 4.2.5,

$r(4,4) < r(3,4) + r(4,3)$

$= 9 + 9,$ by Problems 1 and 3.

$= 18.$ Thus, $r(4,4) < 18.$

But the graph on 17 vertices shown in Figure 4.2.4 has no clique of 4 vertices and no independent set of 4 vertices.

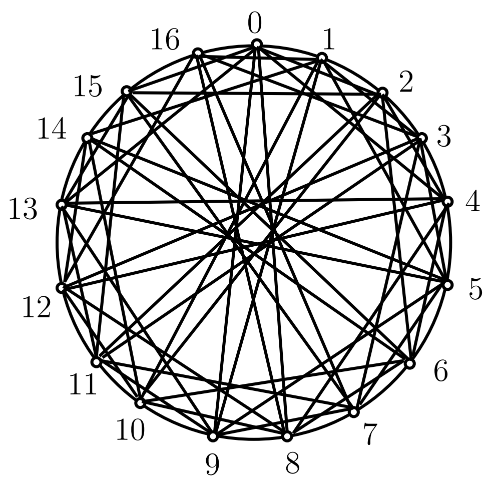


Figure 4.2.4

Thus, $r(4,4) \geq 18$ ...(2)

From (1) and (2), $r(4,4) = 18$.

The following table shows all Ramsey numbers $r(k,l)$ known to date.

k *l*	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	18	23
4	1	4	9	18			

Definition 4.2.6. A (k,l)*-Ramsey graph* is a graph on $r(k,l)$ vertices that contains neither a clique of k vertices nor an independent set of l vertices.

By definition of $r(k,l)$, such graphs exist for all $k \geq 2$ and $l \geq 2$. The graph in Figures $4.2.1$ to $4.2.4$ are Ramsey graphs.

The next theorem provides an upper bound for $r(k,l)$.

Theorem 4.2.7. $r(k,l) \leq \binom{k+l-2}{k-1}$.

Proof. We prove by induction on $k+l\ (\geq 2)$.

If $k+l=2$, then $k=l=1$ and since $r(1,1)=1=\binom{0}{0}$. If $k+l=3$, then one of k and l, say $k=2$ and hence $r(2,l)=r(2,1)=1=\binom{1}{1}$. If $k+l=4$, then $r(1,3)=1=\binom{2}{0}$ and $r(2,2)=2=\binom{2}{1}$. If $k+l=5$, then $r(1,4)=1=\binom{3}{0}$, $r(2,3)=3=\binom{3}{1}$ and $r(2,2)=2=\binom{2}{1}$. Thus, we see that the theorem holds when $k+l \leq 5$.

Let m and n be positive integers. Assume that the theorem holds for all positive integers k and l such that $5 \leq k+l < m+n$. Then by Theorem 4.2.5, we have

$$\begin{aligned} r(m,n) &\leq r(m,n-1)+r(m-1,n) \\ &\leq \binom{m+n-3}{m-1}\binom{m+n-3}{m-2} \text{ by induction assumption.} \\ &\leq \binom{m+n-2}{m-1} \end{aligned}$$

Therefore, the theorem holds for all values of k and l. □

In 1947, Erdos has given a lower bound for $r(k,l)$.

Theorem 4.2.8. $r(k,k) \geq 2^{\frac{k}{2}}$.

Proof. Since $r(2,2) = 2$, we may assume that $k \geq 3$. Let $\mathcal{G}_n$ denote the set of all simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $\mathcal{G}_n^k$ denote those graphs in $\mathcal{G}_n$ that have a clique of k vertices. Clearly,

$$|\mathcal{G}_n| = 2^{\binom{n}{2}} \qquad \ldots(1)$$

Since each subset of the $\binom{n}{2}$ possible edges v_iv_j determines a graph in $\mathcal{G}_n$. Similarly, the number of graphs in $\mathcal{G}_n$ having a particular set of k vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct k-element subsets of $\{v_1, v_2, \ldots, v_n\}$, we have

$$|\mathcal{G}_n^k| \leq \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}} \qquad \ldots(2)$$

From (1) and (2),

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} \leq \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k \ 2^{-\binom{k}{2}}}{k!} \qquad \ldots(3)$$

Suppose, now that $n < 2^{\frac{k}{2}}$.

From (3), it follows that

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} \leq \frac{2^{\frac{k^2}{2}} \ 2^{-\binom{k}{2}}}{k!} = \frac{2^{\frac{k}{2}}}{k!} \leq \frac{1}{2}$$

Therefore, fewer than half of the graphs in $\mathcal{G}_n$ contain a clique of k vertices. Also, because $\mathcal{G}_n = \{G \mid G^c \in \mathcal{G}_n\}$, fewer than half of the graphs in $\mathcal{G}_n$ contain an independent set of k vertices. Hence some graph in $\mathcal{G}_n$ contains neither a clique nor an independent set of k vertices. Because this holds for any $n < 2^{\frac{k}{2}}$, we have $r(k,k) \geq 2^{\frac{k}{2}}$. □

Corollary 4.2.9. *If* $m = min\{k,l\}$, *then* $r(k,l) \geq 2^{\frac{m}{2}}$.

$r(k,l)$ can be thought of as the smallest integer n such that every 2-edge colouring (E_1,E_2) of K_n contains either a complete subgraph on k vertices, all of whose edges are in colour 1 or a complete subgraph on l vertices, all of whose edges are in colour 2. Expressed in this form, the Ramsey number have a natural generalisation. We define $r(k_1,k_2,\ldots,k_m)$ to be the smallest integer such that every m-edge colouring $(E_1,E_2,\ldots,E_m)$ of K_n contains for some $i,$ a complete subgraph on K_i vertices, all of whose edges are in colour $i.$

The following theorem and corollary generalise Ramsey theorem and the upper bound theorem.

Theorem 4.2.10. $r(k_1,k_2,\ldots,k_m) \leq r(k_1-1,k_2,\ldots,k_m)+r(k_1,k_2-1,\ldots,k_m)+\ldots+r(k_1,k_2,\ldots,k_m-1)-m+2.$

Corollary 4.2.11. $r(k_1+1,k_2+1,\ldots,k_m+1) \leq \dfrac{(k_1+k_2+\ldots+k_m)!}{k_1!k_2!\ldots,k_m!}.$

Exercises

1. If G and H are two simple graphs, prove that $\alpha(G[H]) \leq \alpha(G)\alpha(H).$
2. Prove that $r(kl+1,kl+1)-1 \geq (r(k+1,k+1)-1)\times(r(l+1,l+1)-1).$
3. Prove that $r(2^n+1,2^n+1) \geq 5^n+1$ for all $n \geq 0.$
4. Prove that the join of a 3-cycle and a 5-cycle contains no $K_6,$ but that every 2-edge colouring yields a monochromatic triangle.

4.3 Turan's Theorem

In this section, we prove a well-known theorem due to Turan (1941), which determines the maximum number of edges that a simple graph on ν vertices and not containing a clique of size $m+1$ can have. Turan's theorem has become the basis of a significant branch theory known as extremal graph theory. We derive it from the following theorem of Erdos (1970).

Theorem 4.3.1. (***Erdos***) *If a simple graph G contains no $K_{m+1},$ then G is degree majorised by some complete m-partite graph $H.$ Moreover, if G has the same degree sequence as $H,$ then $G \cong H.$*

Proof. By induction on $m.$ The theorem is trivial for $m=1.$ Assume that it holds for all $m < n$ and let G be a simple graph which contains no K_{n+1}. Choose a vertex u of degree Δ in G and set $G_1 = G[N(u)]$. Since G contains no K_{n+1}, G_1 contains no K_n and therefore, by induction hypothesis, G_1 is degree majorised by some complete $(n-1)$ -partite graph H_1.

Now, set $V_1 = N(u)$ and $V_2 = V \backslash V_1$ and denote by G_2 the graph whose vertex set is V_2 and whose edge set is empty. Consider the join $G_1 \vee G_2$ of G_1 and G_2. Since

$N_G(v) \subseteq N_{G_1 \vee G_2}(v)$ for all $v \in V_1 \ldots\ldots (1)$

Since each vertex of V_2 has degree Δ in $G_1 \vee G_2$, G is degree majorised by $G_1 \vee G_2$. Therefore, G is also degree majorised by the complete n -partite graph $H = H_1 \vee G_2$.

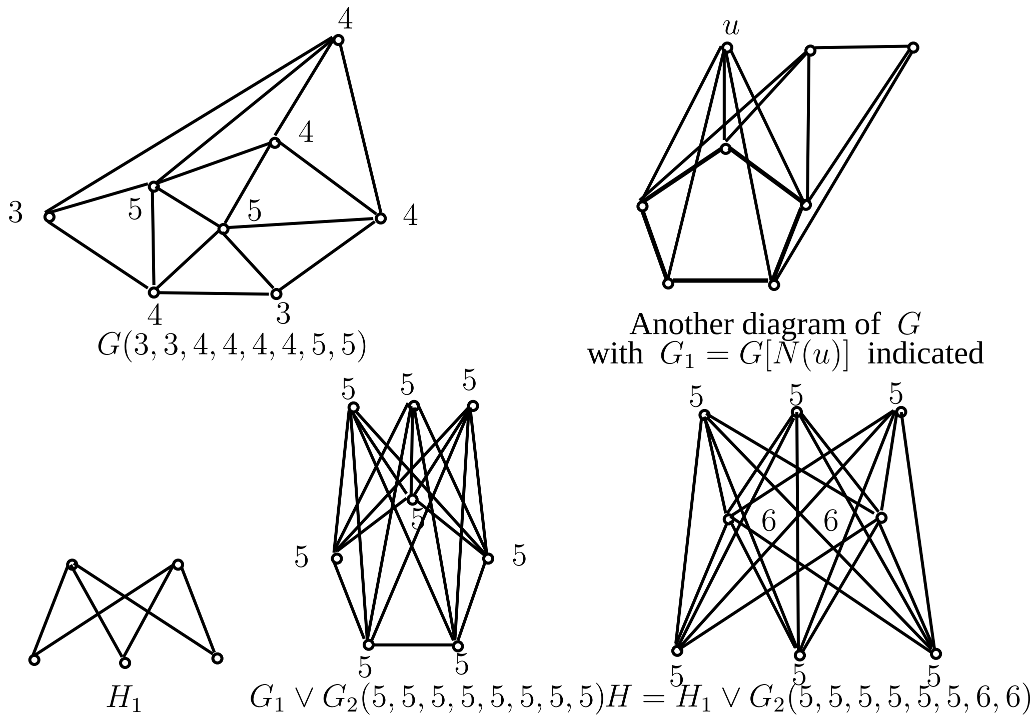


Figure 4.3.1

Suppose now that G has the same degree sequence as H, then G has the same degree sequence as $G_1 \vee G_2$ and hence equality must hold in (1). Thus, in G, every vertex of V_1 must be joined to every vertex of V_2. It follows that $G = G_1 \vee G_2$. Since $G = G_1 \vee G_2$ has the same degree sequence as $H = H_1 \vee G_2$, the graphs G_1 and H_1 must have same degree sequence and therefore, by induction hypothesis, they must be isomorphic.

Thus, $G \cong H$. $\square$

Definition 4.3.2. A k *-partite graph* is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset. A complete k -partite graph is one that is simple and in which each vertex is joined to every vertex that is not in the same subset. The complete m -partite graph on n vertices is one in which each part has either $[\frac{n}{m}]$ or $\{\frac{n}{m}\}$ vertices and is denoted by $T_{m,n}$. That is, $T_{m,n}$ is the complete m -partite graph on n vertices in which all parts are as equal in size as possible. The graph H shown in Figure 4.3.1 is $T_{3,8}$.

Theorem 4.3.3. $($***Turan***$)$ *If* G *is a simple graph and contains no* K_{m+1}, *then* $\epsilon(G) \leq \epsilon(T_{m,\nu})$. *Moreover,* $\epsilon(G) = \epsilon(T_{m,\nu})$ *only if* $G \cong T_{m,\nu}$.

Proof. Let G be a simple graph that contains no K_{m+1}. Therefore, by Theorem **??**, G is degree majorised by some complete m -partite graph H.

Obviously, $\epsilon(G) \leq \epsilon(H)$ ...(1)

But $\epsilon(H) \leq \epsilon(T_{m,\nu})$ ...(2)

From (1) and (2), $\epsilon(G) \leq \epsilon(T_{m,\nu})$, proving the first part.

Suppose $\epsilon(G) = \epsilon(T_{m,\nu})$, then equality must hold in (1) and (2). Since $\epsilon(G) = \epsilon(H)$ and G is degree majorised by H, G must have the same degree sequence as H.

Therefore, by Theorem 4.3.1, $G \cong H$.

Also, since $\epsilon(H) = \epsilon(T_{m,\nu})$, it follows that $H \cong T_{m,\nu}$.

We conclude that $G \cong T_{m,\nu}$. □

Solved Problems

Problem 1. Prove that in any set of six people, there will always be either of three who are mutually acquainted or three who are mutually strangers.

Solution. Consider a graph G on six vertices in which the vertices represent the 6 people and two vertices are adjacent if the corresponding persons are acquainted. Then it is enough to prove that G has three vertices which are adjacent to each other or has three vertices which are not adjacent to each other. In other words, we have to prove that G or G^c contains a triangle.

Let v be a vertex of G. Since G contains 5 vertices other than v, it must be either adjacent to three vertices in G or nonadjacent to three vertices in G. Hence, v must be adjacent to three vertices in G or G^c. Without loss of generality, let us assume that v is adjacent to three vertices in G. If two of these vertices are adjacent, then G contains a triangle. If not, then those three vertices forms a triangle in G^c.

Hence G or G^c contains a triangle.

Exercises

1. In a group of nine people, one person knows two of the others, two people knows four others, four each knows five others and the remaining two each know six others. Show that there are three people who know one another.
2. A certain bridge has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them arrives. Show that at least one more game can now be played.
3. If G is simple, and $\epsilon \geq \frac{\nu^2}{4},$ prove that G contains a triangle.

4.4 Vertex Colourings

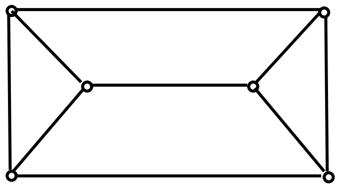
In the previous chapter, we have studied edge colouring of graphs. We now turn our attention to the analogous concept, namely vertex colouring.

Definition 4.4.1. A k *-vertex colouring* of G is an assignment of k colours $1, 2, \ldots, k$ to the vertices of $G.$ The colouring is said to be proper if no two distinct adjacent vertices have the same colour. Thus a proper k -vertex colouring of a loopless graph G is a partition of V into k independent (possibly empty) sets $(V_1, V_2, \ldots, V_k).$ G is k -vertex colourable if G has a proper k -vertex colouring.

Terminology.*For convenience, we shall abbreviate ”proper vertex colouring” as simply a ”colouring” and ”proper k -vertex colouring” as simply a ” k -colouring” and ” k -vertex colourable” as ” k -colourable”.*

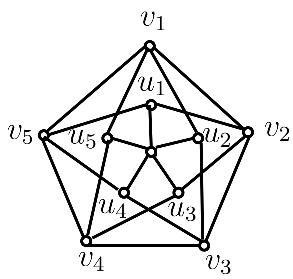
Remark 4.4.2. (i) G is k -colourable if and only if its underlying graph is k -colourable. Hence, **in discussing colourings, we shall restrict ouselves to simple graphs.**
(ii) G is 1 -colourable if and only if it is empty.
(iii) G is 2 -colourable if and only if it is bipartite.

Definition 4.4.3. The *chromatic number* $\chi(G)$ of a graph G is the minimum k for which G is k -colourable. If $\chi(G) = k,$ G is said to be k -chromatic. A 3 -chromatic graph is shown in Figure $4.4.1.$



3-chromatic graph
Figure $4.4.1$

Definition 4.4.4. A graph G is *critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of $G.$ A k -critical graph is one that is k -chromatic and critical. A 4 -critical graph is shown in Figure $4.4.1.$



Grotzsch graph - a 4-critical graph
Figure $4.4.2.$

Remark 4.4.5. (i) Every k -chromatic graph has a k -critical subgraph.
(ii) Every critical graph is connected.

Theorem 4.4.6. *If G is k -critical, then $\delta \geq k-1.$*

Proof. We prove by the method of contradiction.
If possible, let G be a k -critical graph with $\delta < k-1.$ Let v be a vertex of degree δ in $G.$

Since G is k -critical, $G-v$ is $(k-1)$ -colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a $(k-1)$ -colouring of $G-v.$

By definition, v is adjacent in G to $\delta < k-1$ vertices and therefore, v must be adjacent in G to every vertex of some $V_j.$ But then, $(V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{k-1})$ is a $(k-1)$ -colouring of $G,$ a contradiction to G is k -critical.
Thus, $\delta \geq k-1.$ $\square$

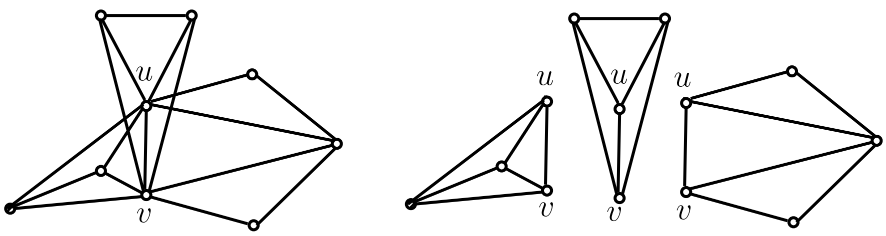
Corollary 4.4.7. *Every k -chromatic graph has at least k vertices of degree at least $k-1.$*

Proof. Let G be a k-chromatic graph and let H be a k-critical subgraph of G. By Theorem 4.4.6, each vertex of H has degree at least $k-1$ in H and hence, also in G. Being a k-chromatic graph, G has at least k vertices of degree at least $k-1$. □

Corollary 4.4.8. *For any graph* G, $\chi \leq \Delta + 1$.

Proof. Suppose $\chi \geq \Delta + 2$. Then, by Corollary 4.4.7, G has at least χ vertices of degree at least $\chi - 1 \geq \Delta + 1$, which is impossible. □

Definition 4.4.9. Let S be a vertex cut of a connected graph G and let the components of $G - S$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the subgraph $G_i = G[V_i \cup S]$ are called the S*-components* of G. For $S = \{u, v\}$, the S-components are given in Figure 4.4.3. We say that the colourings $G_1, G_2, \ldots, G_n$ *agree on* S if, for every $v \in S$, vertex v is assigned the same colour in each of the colourings.



G $\{u, v\}$-componenets of G

Figure 4.4.3

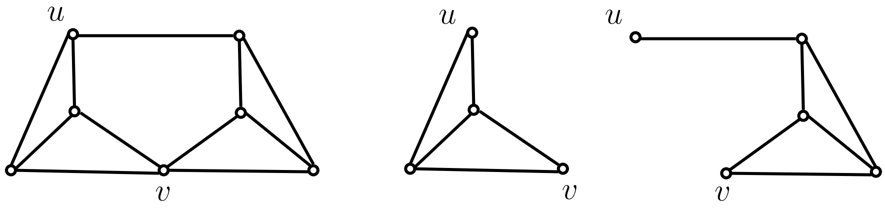
Theorem 4.4.10. *In a critical graph, no vertex cut is a clique.*

Proof. By contradiction. Let G be a k-critical graph and suppose that G has a vertex cut S that is a clique. Denote the S-components of G by $G_1, G_2, \ldots, G_n$. Since G is k-critical, each G_i is $(k-1)$ colourable. Furthermore, since G is a clique, the vertices in S must receive distinct colours in any $(k-1)$-colouring of G_i. It follows that there are $(k-1)$-colourings of $G_1, G_2, \ldots, G_n$ which agree on S. But these colourings together yield a $(k-1)$-colouring of G, a contradiction.
Hence, in a critical graph, no vertex cut is a clique. □

Corollary 4.4.11. *Every critical graph is a block.*

Proof. If v is a cut vertex, then $\{v\}$ is a vertex cut, which is also, trivially a clique. It follows from Theorem4.4.10 that no critical graph has a cut vertex; equivalently, every critical graph is a block. □

Definition 4.4.12. If a critical graph G has a 2-vertex cut $\{u, v\}$, then u and v cannot be adjacent. We say that a $\{u, v\}$-component G_i of G is of *type 1* if every $(k-1)$-colouring of G_i assigns the same colour to u and v, and of *type 2* if every $(k-1)$-colouring of G_i assigns different colours to u and v.



A graph G Type 1 Type 2

Figure $4.4.4$

Theorem 4.4.13. *Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then*
(i) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of G of type i $(i = 1, 2)$, and
(ii) both $G_1 + uv$ and $G_2.uv$ are k-critical graphs (where $G_2.uv$ denotes the graph obtained from G_2 by identifying u and v).

Proof. (i) Since G is critical, each $\{u, v\}$-component of G is $(k-1)$-colourable. Now there cannot exist $(k-1)$-colourings of these $\{u, v\}$-components all of which agree on $\{u, v\}$, since such colourings would together yield a $(k-1)$-colouring of G. Therefore, there are two $\{u, v\}$-components G_1 and G_2 such that no $(k-1)$-colouring of G_1 agrees with any $(k-1)$-colouring of G_2. Clearly, one, say G_1, must be of type 1 and the other, G_2, of type 2. Since G_1 and G_2 are of different types, the subgraph $G_1 \cup G_2$ of G is not $(k-1)$-colourable. Therefore, because G is critical, we must have $G = G_1 \cup G_2$.

(ii) Set $H_1 = G_1 + uv$. Since G_1 is of type 1, H_1 is k-chromatic. We shall prove that H_1 is critical by showing that, for every edge e of H_1, $H_1 - e$ is $(k-1)$-colourable. This is clearly so if $e = uv$, since then $H_1 - e = G_1$. Let e be some other edge of H_1. In any $(k-1)$-colouring of $G - e$, the vertices u and v must receive different colours, since G_2 is a subgraph of $G - e$. The restriction of such a colouring to the vertices of G_1 is a $(k-1)$-colouring of $H_1 - e$. Thus $G_1 + uv$ is k-critical. An analogous argument shows that $G_2.uv$ is k-critical. □

Corollary 4.4.14. *Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then* $d(u) + d(v) \geq 3k - 5$.

Proof. Let G_1 be a $\{u, v\}$-component of type 1 and G_2 be a $\{u, v\}$-component of type 2. Set $H_1 = G_1 + uv$ and $H_2 = G_2.uv.$ By Theorem 4.4.13 and the fact that $\delta \geq k-1,$ we have

$$d_{H_1}(u) \geq k-1 \text{ and } d_{H_1}(v) \geq k-1, \text{ since } \delta \geq k-1.$$

Therefore, $d_{H_1}(u) + d_{H_1}(v) \geq 2k-2,$ and

$$d_{H_2}(w) \geq k-1,$$

where, w is the new vertex obtained by identifying u and $v.$
It follows that

$$d_{G_1}(u) + d_{G_1}(v) \geq 2k-2-2$$
$$\Rightarrow d_{G_1}(u) + d_{G_1}(v) \geq 2k-4 \ldots\ldots (1) \text{ and}$$
$$d_{G_2}(u) + d_{G_2}(v) \geq d_{H_2}(w)$$
$$\Rightarrow d_{G_2}(u) + d_{G_2}(v) \geq k-1 \ldots\ldots (2)$$

Inequalities in (1) and (2) yields

$$\begin{aligned} d_G(u) + d_G(v) &\geq 2k-4+k-1 \\ &\geq 3k-5. \end{aligned}$$

$\square$

Definition 4.4.15. A graph G is uniquely k -colourable if any two k -colourings of G induce the same partition of $V.$ The cycle C_4 is uniquely 2-colourable as shown in Figure $4.4.5.$

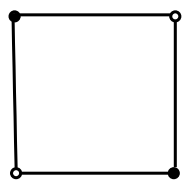


Figure $4.4.5$

Solved Problems

Problem 1. Prove that for any graph $G,$ the following are equivalent

(a) G is 2-colourable.

(b) G is bipartite.

(c) Every cycle of G has even length.

Solution. (a) $\Rightarrow$ (b): Assume that G is 2-colourable. Then $V(G)$ can be partitioned into two independent sets (colour classes). Hence they form a bipartition of $G.$

(b) $\Rightarrow$ (c): Assume that G is bipartite. Then G contains no odd cycle. That is, every cycle of G has even length.

(c) $\Rightarrow$ (a): Assume that every cycle of G has even length. Then G is bipartite. Hence $V(G)$ can be partitioned into two sets V_1 and V_2 such that both are independent sets in G. Then the vertices of V_1 can be coloured with one colour and the vertices of V_2 can be coloured with another colour. Thus, (V_1, V_2) is a 2-colouring of G. Hence, G is 2-colourable. ■

Problem 2. If G is uniquely k-colourable, prove that $\delta(G) \geq k-1$.
Solution. Let v be any vertex of G. In any k-colouring of G, v must be adjacent with at least one vertex of every colour, different from the one that is assigned to v. Otherwise, by re-colouring v with a colour which none of its neighbour is having, we get a different k-colouring. Hence $d(v) \geq k-1$ and hence $\delta(G) \geq k-1$. ■

Problem 3 If G is uniquely k-colourable, prove that in any k-colouring of G, the subgraph induced by the union of any two colour classes is connected.
Solution. Let $(C_1, C_2, \ldots, C_k)$ be any k-colouring of G. Consider two classes, say C_1 and C_2. Suppose $C_1 \cup C_2$ is disconnected. Let H be a component of the subgraph induced by $C_1 \cup C_2$. Obviously, no vertex of H is adjacent to a vertex of $V(G)-V(H)$ that is coloured either C_1 or C_2. Hence, interchanging the colour of the vertices in H and retaining the original colours for all other vertices, we get a different k-colouring of G, which is a contradiction to G is uniquely k-colourable.
Hence $C_1 \cup C_2$ is connected. ■

Problem 4 Prove that every uniquely k-colourable graph is $(k-1)$-connected.
Solution. Let G be a uniquely k-colourable graph. Suppose G is not $(k-1)$-connected. Then there exists a set S of at most $k-2$ vertices such that $G-S$ is either trivial or disconnected. If $G-S$ is trivial, then G has at most $n-1$ vertices so that G is not uniquely k-colourable. Hence $G-S$ has at least two components and there are at least two colours c_1 and c_2 that are assigned to any vertices of S.

If every vertex in a component of $G-S$ has colour different from c_1 and c_2, then by assigning colour c_1 to a vertex of this component, we get a different k-colouring of G. Otherwise, by interchanging the colours c_1 and c_2 in a component of $G-S$, a different k-colouring of G is obtained. In any case, G is not uniquely k-colourable, which is a contradiction. Hence G is $(k-1)$-connected. ■

Problem 5 Let G be a uniquely k-colourable graph. Prove that for any m, where $2 \leq m \leq k$, the subgraph induced by the union of any m colour classes is $(m-1)$-connected in any k-colouring of G.
Solution. Let H be the subgraph induced by the union of any m colour classes. We claim that H is uniquely m-colourable. Suppose H has different m-colourings. Then

these m -colourings of H induce different k -colourings in $G,$ which is a contradiction. Thus H is uniquely m -colourable. Hence by Problem 4, H $(m-1)$ -connected.

Exercises

1. If G is simple, prove that $\chi \geq \dfrac{\nu^2}{\nu^2 - 2\epsilon}.$
2. If any two odd cycles of G have a vertex in common, prove that $\chi \leq 5.$
3. Show that $\chi(G) \leq 1 + max\ \delta(H),$ where the maximum is taken over all induced subgraphs H of $G.$
4. Show that the only 1 -critical graph is K_1 and the only 2 -critical graph is $K_2.$
5. Show that the only 3 -critical graphs are the odd cycles.
6. Show that no vertex cut of a k -critical graph induces a uniquely $(k-1)$ -colourable subgraph.
7. Show that if u and v are vertices of a critical graph, then $N(u)$ is not a subset of $N(v).$
8. Prove that no k -critical graph has exactly $k+1$ vertices.(Hint: use Exer.7.)
9. Show that $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2).$
10. Prove that $G_1 \vee G_2$ is critical if and only if both G_1 and G_2 are critical.
11. For $n = 4$ and all $n \geq 6,$ construct a 4-critical graph on n vertices.

4.5 Brooks' Theorem and Hajos' Conjecture

The following theorem due to Brooks (1941) shows that there are only two types of graphs for which $\chi = \Delta + 1.$

Theorem 4.5.1. *If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta.$*

Proof. Let G be a connected simple graph and is neither an odd cycle nor a complete graph. Let $\chi(G) = k.$
Since every k - chromatic graph contains k -critical subgraph, without loss of generality, we may assume that G is k -critical. By Corollary 4.4.11, G is a block. Since 1 -critical and 2 -critical graphs are complete and 3 -critical graphs are odd cycles, we have $k \geq 4.$

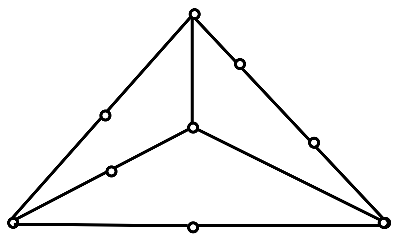
If G has a connectivity 2, then there is a 2-vertex cut $\{u, v\}$ in $G.$ Then

$$2\Delta \geq d(u) + d(v) \geq 3k - 5 = (2k-1) + (k-4) \geq 2k - 1.$$

Since 2Δ is even, $\chi = k \leq \Delta.$

Now assume that G is 3-connected. Since G is not complete, there exist vertices u, v and w in G such that $uv, vw \in E$ and $uw \notin E.$ Set $u = v_1$ and $w = v_2$ and let $v_3, v_4, \ldots, v_\nu = v$ be any ordering of the vertices of $G - \{u, w\}$ such that each v_i is adjacent to some v_j with $j > i.$ (This can be achieved by arranging the vertices of $G - \{u, w\}$ in nonincreasing order of their distance from $v.$) We can now describe a Δ -colouring of G by assigning colour 1 to $v_1 = u$ and $v_2 = w;$ then successively colour $v_3, v_4, \ldots, v_\nu = v,$ each with the first available colour in the list $1, 2, \ldots, \Delta.$ By the construction of the sequence $v_1, v_2, \ldots, v_\nu,$ each vertex $v_i, 1 \leq i \leq \nu - 1,$ is adjacent to some vertex v_i with $j > i$ and therefore to at most $\Delta - 1$ vertices v_j with $j < i.$ It follows that, when its turn comes to be coloured, v_i is adjacent to at most $\Delta - 1$ colours and thus one of the colours $1, 2, \ldots, \Delta$ will be available. Finally, since v_ν is adjacent to two vertices of colour 1 (namely v_1 and v_2), it is adjacent to at most $\Delta - 2$ other colours and can be assigned one of the colours $2, 3, \ldots, \Delta.$ $\square$

Definition 4.5.2. A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivision. Figure $4.5.1$ shows the subdivision of $K_4.$



A subdivision of K_4

Figure $4.5.1$

If G is k -chromatic, then G contains a subdivision of $K_k.$ This is known as *Hajos' conjecture.* Note that, a 4-cycle is a subdivision of $K_3,$ but is not 3-chromatic.

For $k = 1$ and $k = 2,$ the validity of Hajos' conjecture is obvious. It can also be verified for $k = 3,$ because a 3-chromatic graph necessarily contains an odd cycle and every odd cycle is a subdivision of $K_3.$ Dirac settled the case for $k = 4.$

Theorem 4.5.3. *If G is 4 -chromatic, then G contains a subdivision of $K_4.$*

Proof. Let G be a 4-chromatic graph. Since every k - chromatic graph contains k -

critical subgraph and if some subgraph of G contains a subdivision of $K_4,$ then does $G,$ without loss of generality, we may assume that G is k -critical. By Corollary 4.4.11, G is a block with $\delta \geq 3$ and so $\nu \geq 4.$
We proceed by induction on $\nu.$ If $\nu = 4,$ then G is K_4 and the theorem holds obviously. Assume that the theorem holds for all graphs with fewer than n vertices and let $\nu(G) = n > 4.$

Suppose G is 2-connected and $\{u, v\}$ is a 2-vertex cut of $G.$ By Theorem 4.4.13, G has two components G_1 and $G_2,$ where $G_1 + uv$ is 4-critical. Since $\nu(G_1 + uv) < \nu(G),$ we can apply the induction hypothesis and deduct that $G_1 + uv$ contains a subdivision of $K_4.$ It follows that, if P is a (u, v) -path in $G_2,$ then $G_1 \cup P$ contains a subdivision of $K_4.$ Since $G_1 \cup G_2 \subseteq G,$ G contains a subdivision of $K_4.$

Suppose G is 3-connected. Since $\delta \geq 3,$ G has a cycle C of length at least four. Let u and v be nonconsecutive vertices on $C.$ Since $G - \{u, v\}$ is connected, there is a path P in $G - \{u, v\}$ connecting the two components of $C - \{u, v\}.$ We assume that the origin x and the terminus y are the only vertices of P on $C.$ Similarly, there is a path Q in $G - \{x, y\}.$

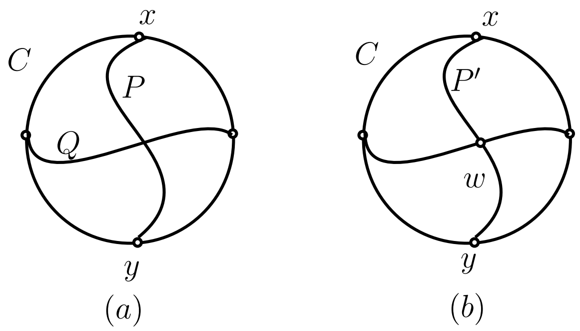


Figure $4.5.2$

If P and Q have no vertex in common, then $C \cup P \cup Q$ is a subdivision of $K_4.$ Otherwise, let w be the first vertex of P on Q and let P' denote the (x, w) -section of $P.$ Then $C \cup P' \cup Q$ is a subdivision of $K_4.$ Hence in both cases G contains a subdivision of $K_4.$ $\square$

Hajos' conjecture has not yet been settled in general. There is a related conjecture due to Hadwiger (1943): if G is k -chromatic, then G is 'contractible' to a subgraph which contains $K_k.$

4.6 Chromatic Polynomials

We denote the number of distinct k-colourings of G by $\pi_k(G);$ thus $\pi_k(G) > 0$ if and only if G is k-colourable. Two colourings are said to be regarded as distinct if some vertex is assigned different colours in the two colourings.
If $\epsilon(G) = 0,$ then each vertex can be independently assigned any one of the k available colours. Therefore, $\pi_k(G) = k^\nu.$ If G is complete, then there are k choices for the first vertex, $k-1$ choices for the second, $k-2$ choices for the third, and so on. Thus, in this case, $\pi_k(G) = k(k-1)(k-2)\ldots(k-\nu+1).$
A triangle has six distinct 3-colourings.

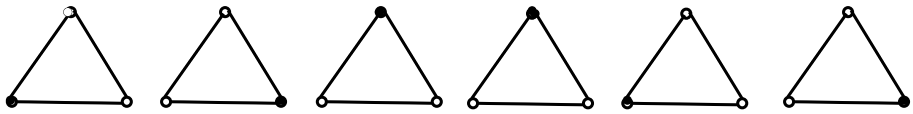


Figure $4.6.1$

Note that even though there is exactly one vertex of each colour in each colouring, we still regard these six colourings as distinct.

Theorem 4.6.1. *If G is simple, then $\pi_k(G) = \pi_k(G-e) - \pi_k(G.e)$ for any edge e of $G.$*

Proof. Let u and v be the ends of $e.$ To each k-colouring of $G-e$ that assigns the same colour to u and $v,$ there corresponds a k-colouring of $G.e$ in which the vertex of $G.e$ formed by identifying u and v is assigned the common colour of u and $v.$ This correspondence is clearly a bijection. Therefore, $\pi_k(G.e)$ is precisely the number of k-colourings of $G-e$ in which u and v are assigned the same colour.

Also, since each k-colouring of $G-e$ that assigns different colour to u and v is a k-colouring of G and conversely, $\pi_k(G)$ is the number of k-colourings of $G-e$ in which u and v are assigned different colours. Hence it follows that $\pi_k(G-e) = \pi_k(G) + \pi_k(G.e)$ □

Corollary 4.6.2. *For any graph $G,$ $\pi_k(G)$ is a polynomial in k of degree $\nu,$ with integer coefficients, leading term k^ν and constant term zero. Furthermore, the coefficients of $\pi_k(G)$ alternate in sign.*

Proof. By induction on $\epsilon.$ We may assume, without loss of generality that G is simple. If $\epsilon = 0$ then $\pi_k(G) = k^\nu,$ which trivially satisfies the conditions of the corollary.

Suppose, now, that the corollary holds for all graphs with fewer than m edges and let G be a graph with m edges, where $m \geq 1$. Let e be any edge of G. Then both $G - e$ and $G.e$ have $m - 1$ edges and it follows from the induction hypothesis that there are nonnegative integers $a_1, a_2, \ldots, a_{\nu-1}$ and $b_1, b_2, \ldots, b_{\nu-2}$ such that

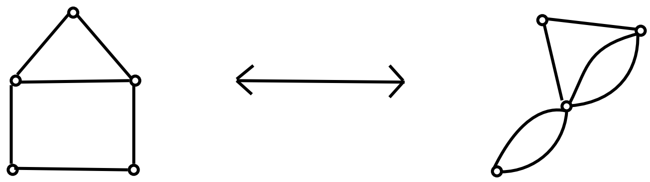


Figure 4.6.2

$$\pi_k(G - e) = \sum_{i=1}^{\nu-1} (-1)^{\nu-i} a_i k^i + k^\nu \text{ and}$$
$$\pi_k(G.e) = \sum_{i=1}^{\nu-2} (-1)^{\nu-1-i} a_i k^i + k^{\nu-1}$$

By the above theorem,

$$\begin{aligned} \pi_k(G) &= \pi_k(G - e) - \pi_k(G.e) \\ &= \sum_{i=1}^{\nu-2} (-1)^{\nu-i} (a_i + b_i) k^i - (a_{\nu-1} + 1) k^{\nu-1} + k^\nu. \end{aligned}$$

Thus, G too satisfies the conditions of the corollary. Hence the result follows by induction hypothesis. □

Corollary 4.6.3. *If G is simple, then the coefficient of $k^{\nu-1}$ in $\pi_k(G)$ is $-\epsilon$.*

Proof. We shall prove the result by induction on ϵ.
If $\epsilon = 0$, then $\pi_k(G) = k^\nu$. Hence the coefficient of $k^{\nu-1} = 0 = -\epsilon$.
Now assume the result for all graphs with less than ϵ edges. Let e be an edge of G.
Then, by Theorem 4.6.1, $\pi_k(G) = \pi_k(G - e) - \pi_k(G.e)$ ...(1)
Since $G - e$ has $\epsilon - 1$ edges (less than ϵ edges), by induction hypothesis, the coefficient of $k^{\nu-1}$ in $\pi_k(G - e)$ is $-\epsilon$.
Hence the coefficient of $k^{\nu-1}$ in $\pi_k(G) = -(\epsilon - 1) - 1$, using (1)
$= -\epsilon$. □

We can refer to the function $\pi_k(G)$ as the *chromatic polynomial* of G. Chromatic polynomial can be calculated in either of the two ways.

(i) by repeatedly applying the recursion $\pi_k(G) = \pi_k(G-e) - \pi_k(G.e)$ and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of empty graphs, or

(ii) by repeatedly applying the recursion $\pi_k(G-e) = \pi_k(G) + \pi_k(G.e)$ and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of complete graphs.

Method (i) is more suited for graphs with few edges whereas (ii) can be applied more efficiently to graphs with many edges.

Solved Problems

Problem 1. If $G_1, G_2, \ldots, G_\omega$ are components of G, prove that $\pi_k(G)$
$= \pi_k(G_1)\pi_k(G_2)\ldots\pi_k(G_\omega)$.

Solution. Number of ways of colouring G_1 with k colours is $\pi_k(G_1)$. Since any choice of k -colourings for $G_1, G_2, \ldots, G_\omega$ can be combined to give a k -colouring, we have $\pi_k(G) = \pi_(G_1)\pi_(G_2)\ldots\pi_(G_\omega)$. ∎

Problem 2 A simple graph G on n vertices is a tree if and only if $\pi_k(G) = k(k-1)^{n-1}$.

Solution. Let G be a tree. We prove that $\pi_k(G) = k(k-1)^{n-1}$ by induction on n.
If $n = 1$, then the result is trivial. So we assume that the result holds for all trees with at most $n-1$ vertices, $n \geq 2$.

Let G be a tree on n vertices and e be a pendant edge of G. By Theorem 4.6.1, $\pi_k(G) = \pi_k(G-e) - \pi_k(G.e)$. Now $G-e$ is a forest with two components of order $(n-1)$ and 1. Hence, $\pi_k(G-e) = (k(k-1)^{n-1})k$, by Problem 1.
Since $G.e$ is a forest with $(n-1)$ vertices, $\pi_k(G.e) = k(k-1)^{n-2}$
Thus, $\pi_k(G-e) = (k(k-1)^{n-1})k - k(k-1)^{n-2}$
$= k(k-1)^{n-1}$
Conversely, assume that G is a simple graph with $\pi_k(G) = k(k-1)^{n-1}$
$= k^n - (n-1)k^{n-1} + \ldots + (-1)^{n-1}k.$
Hence, by Corollaries 4.6.2 and 4.6.3, G has n vertices and $(n-1)$ edges. Further, the last term, $(-1)^{n-1}k$ ensures that G is connected. Hence, G is a tree. ∎

Problem 3 Prove that $k^4 - 3k^3 + 3k^2$ cannot be the chromatic polynomial of any graph.

Solution. Suppose that there exists a graph G such that $\pi_k(G) = k^4 - 3k^3 + 3k^2$. Then the number of vertices in G is 4 and the number of edges is 3 (by Problem 2).

Case 1. Suppose G is connected. Since $q = 3 = p - 1$, G is a tree.
Hence, by Problem 2, $\pi_k(G) = k(k-1)^3$
$= k^4 - 3K^3 + 3k^2 - k$, which is a contradiction.

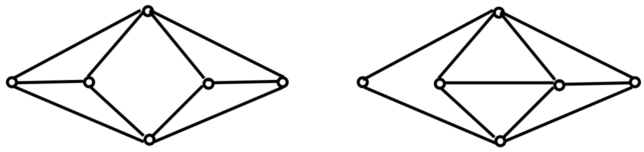
Case 2. Suppose G is not connected.
Then $G = K_3 \cup K_1$
Therefore, $\pi_k(G) = \pi(K_3)\pi(K_1)$
$= k(k-1)(k-2)k$
$= k^4 - 3K^3 + 3k^2,$ which is again a contradiction.

Hence the result is proved.

Remark 4.6.4. Chromatic polynomial of a graph does not fix the graph uniquely up to isomorphism. For example, $k(k-1)$ is the chromatic polynomial of both the nonisomorphic graphs $K_{1,3}$ and P_4.

Exercises

1. Calculate the chromatic polynomials of the following two graphs:



2. If G is a cycle of length $n,$ prove that $\pi_k(G) = (k-1)^n + (-1)^n(k-1).$
3. Show that $\pi_k(G \vee K_1) = k\pi_{k-1}(G).$
4. If G is a wheel with n spokes, then prove that $\pi_k(G) = k(k-2)^2 + (-1)^n k(k-1).$
   (Hint: Use Exercises 1 and 2)
5. If G is complete, prove that $\pi_k(G \cup H)\pi_k(G \cap H) = \pi_k(G)\pi_k(H).$
6. Prove that a graph G is connected if and only if the coefficient of k in $\pi_k(G)$ is not zero.

4.7 Girth and Chromatic Number

In any colouring of a graph, the vertices in clique must all be assigned different colours. Thus a graph with large clique neccessarily has a high chromatic number.

Theorem 4.7.1. *For any positive integer* $k,$ *there exists a* k *-chromatic graph containing no triangle.*

Proof. For $k = 1$ and $k = 2$, the graphs K_1 and K_2 have the required property. We proceed by induction on k. Suppose that we have already constructed a triangle-free graph G_k with chromatic number $k \geq 2$. Let the vertices of G_k be $v_1, v_2, \ldots, v_n$. Form a new graph G_{k+1} from G_k by adding $n+1$ new vertices $u_1, u_2, \ldots, u_n, v$ and then, for $1 \leq i \leq n$, join u_i to the neighbours of v_i and to v. For example, if G_2 is K_2 then G_3 is the 5-cycle and G_4 the Grotzsch graph(see Figure $4.6.3$).

The graph G_{k+1} clearly has no triangles. For, since $\{u_1, u_2, \ldots, u_n\}$ is an independent set in G_{k+1}, no triangles can have more than one u_i and if $u_i v_j v_k u_i$ were a triangle in G_{k+1}, then $u_i v_j v_k u_i$ would be a triangle in G_k, contrary to assumption.

We now show that G_{k+1} is $(k+1)$ -chromatic. Note that G_{k+1} is certainly $(k+1)$ -colourable, since any k -colouring of G_k can be extended to a $(k+1)$ -colouring of G_{k+1} by colouring u_i with the same colour as v_i, $1 \leq i \leq n$, and then assigning a new colour to v. Therefore it remains to show that G_{k+1} is not k -colourable. If possible, consider a k -colouring of G_{k+1} in which, without loss of generality, v is assigned colour k. Now recolour each vertex v_i of colour k with the colour assigned to u_i.

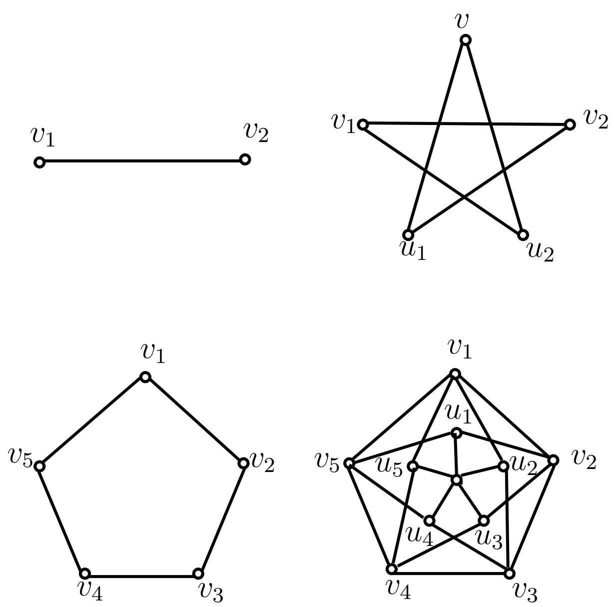


Figure $4.6.3.$ Mycielski' construction

This results in a $(k-1)$ -colouring of k -chromatic graph G_k. Therefore, G_{k+1} is indeed $(k+1)$ -chromatic. Hence the theorem follows from the principle of induction. $\square$

By starting with 2-chromatic graph $K_2,$ the above construction yields, for all $k \geq 2,$ a triangle free k -chromatic graph of order $3.2^{k-2} - 1.$

Using the probablistyic method, Erdos(1961) has shown that, given two integers $k \geq 2$ and $l \geq 2,$ there is a graph with girth k and chromatic number $l.$

Exercises

1. Let $G_3, G_4, \ldots$ be the graphs obtained from $G_2 = K_2,$ using Mycielski's construction, show that each G_k is k -critical.
2. If G is simple with $\nu \geq 4$ and $\epsilon = 2\nu - 2$ that contains no subdivision of $K_4.$
3. For $\nu \geq 4,$ find a simple graph G with $\epsilon = 2\nu - 3$ that contains no subdivision of $K_4.$
4. If G is a graph without loops and $\Delta = 3,$ prove that $\chi \leq 4.$
5. If G is simple and has at most one vertex of degree less than three, prove that G contains a subdivision of $K_4.$

Chapter 5

COMBINATORICS

What is Combinatorics?

Combinatorics is a young field of mathematics, starting to be an independent branch only in the 20th century. However, combinatorial methods and problems have been around ever since. Many combinatorial problems look entertaining or aesthetically pleasing and indeed one can say that roots of combinatorics lie in mathematical recreations and games. Nonetheless, this field has grown to be of great importance in todays world, not only because of its use for other fields like physical sciences, social sciences, biological sciences, information theory and computer science.

Combinatorics is concerned with:

(i) Arrangements of elements in a set into patterns satisfying specific rules, generally referred to as discrete structures. Here discrete (as opposed to continuous) typically also means finite, although we will consider some infinite structures as well.

(ii) The existence, enumeration, analysis and optimization of discrete structures.

(iii) Interconnections, generalizations- and specialization-relations between several discrete structures.

Existence: We want to arrange elements in a set into patterns satisfying certain rules. Is this possible? Under which conditions is it possible? What are necessary, what sufficient conditions? How do we find such an arrangement?

Enumeration: Assume certain arrangements are possible. How many such arrangements exist? Can we say there are at least this many, at most this many or exactly this many? How do we generate all arrangements efficiently?

Classification: Assume there are many arrangements. Do some of these arrangements differ from others in a particular way? Is there a natural partition of all arrangements into specific classes?

Meta-Structure: Do the arrangements even carry a natural underlying structure, e.g., some ordering? When are two arrangements closer to each other or more similar than some other pair of arrangements? Are different classes of arrangements in a particular relation?

Optimization: Assume some arrangements differ from others according to some measurement. Can we find or characterize the arrangements with maximum or minimum measure, i.e. the best or worst arrangements?

5.1 Permutations and Combinations

The words selection and arrangement will be used in the ordinary sense. Thus, there should be no ambiguity in the meanings of statements such as “to select two representatives from five candidates”, ”there are 10 possible outcomes when two representatives are sekcted from five candidates”, “the books are arranged on the shelf”, and “there are 120 ways to arrange five different books on the shelf”. The word combination has the same meaning as the word “selection”, and the word permutation has the same meaning as the word “arrangement”.

Definition 5.1.1. An *r-combination* of n objects is defined as an unordered selection of r of these objects. An $r-$ permutationpermutation of n objects is defined as an ordered arrangement of r of these objects.

For example, to form a committee of 20 senators from 100 senators. It is an unordered selection of 20 senators from the 100 senators and is therefore a 20 combination of the 100 senators. On the other hand, the outcome of a horse race can be viewed as an ordered arrangement of the t horses in the race and is therefore a $t-$ permutation of the t horses.

We are interested here in enumerating the number of combinations or permutations of a given set of objects. Let the notation $C(n, r)$ denote the number of $r-$ combinations of n distinct objects, and the notation $P(n, r)$ denote the number of r -permutations of n distinct objects.

Since there is just one way to select n objects from n objects, $C(n, n) = 1.$ Since there are n ways to select one object out of n objects, $C(n, 1) = n.$

For three objects A, B and $C,$ the selections of two objects are $AB, AC,$ and $BC.$ Hence, $C(3, 2) = 3$ and for three objects $A, B,$ and $C,$ the arrangements of two objects are AB, BA, AC, CA, BC and CB. Hence, $P(3,2) = 6$.

Rule of product: If one event can occur in m ways and another event can occur in n ways, there are $m \times n$ ways in which these two events can occur.

Rule of sum: If one event can occur in m ways and another event can occur in n ways, there are $m,\ m+n$ ways in which one of these two events can occur.

Problem 5.1.2. There are five Roman letters $a, b, c, d,$ and e and three Greek letters α, β and γ.
How many ways are there to select two letters, one from each alphabet?
How many ways are there to select one letter, that is either Roman or Greek?

Solutions (a) By rule of product, there are $5 \times 3 = 15$ ways to select two letters, one from each alphabet.
(b) Since there are five ways to select a Roman letter and three ways to select a Greek letter, by rule of sum, there are $5 + 3 = 8$ ways to select one letter that is either a Roman or a Greek letter.

Remark 5.1.3. Clearly, the occurrence of an event can mean either the selection or the arrangement of a certain number of objects.

Problem 5.1.4. There are five books in Latin, seven books in Greek, and ten books in French. How many ways are there to choose just two books?

Solution (a) By rule of product, 5×7 ways to choose a book in Latin and a book in Greek, 5×10 ways to choose a book in Latin and a book in French, and 7×10 ways to choose a book in Greek. Hence by the rule of sum, and there are $5 \times 7 + 5 \times 10 + 7 \times 10 = 155$ ways to choose two books of different languages.
(b) There are 22×21 = 462 ways to choose two books from the twenty-two books. ■

Result 5.1.5. Using a combinatorial argument, prove that
$$P(n,r) = P(r,r) \times C(n,r).$$

Proof. By the rule of product, one can make an ordered arrangement of r of n distinct objects by first selecting r objects from the n objects and then arranging these r objects in order. Hence, $P(n,r) = P(r,r) \times C(n,r)$. □

Result 5.1.6. Using a combinatorial argument, prove that

$$C(n,r) = C(n-1,r-1) + C(n-1,r).$$

Proof. Suppose that one of the n distinct objects is marked as a special object. The number of ways to select r objects from these n objects is equal to the sum of the number of ways to select r objects so that the special object is always included (there are $C(n-1,r-1)$ such ways] and the number of ways to select r objects so that the special object is always excluded [there are $C(n-1,r)$ such ways). Hence, by the rule of sum $C(n,r) = C(n-1,r-1) + C(n-1,r)$. □

Permutations

Let us now derive an expression for $P(n,r)$, the number of ways of arranging r of n distinct objects. Observe that arranging r of n objects into some order is the same as putting r of the n objects into r distinct (marked) positions. There are n ways to fill the first position (to choose one out of the n objects), $n-1$ ways to fill the second position (to choose one out of the $n-1$ remaining objects),..., and $n-r+1$ ways to fill the last position (to choose one out of the $n-r+1$ remaining objects).
Thus, according to the rule of product, we have

$$P(n,r) = n(n-1)\ldots(n-r+1)$$

Using the notation

$$n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1$$

for $n \geq 1$

$$P(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 3 \times 2 \times 1}{(n-r)\ldots 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

Result 5.1.7. Derive an expression for $P(n,r)$, the number of ways of arranging r of n distinct objects.

Proof. Observe that arranging r of n objects in some order is same as putting r of the n objects into r distinct positions. There are n ways to fill the first position (to choose one out of then objects), $n-1$ ways to fill the second position (to choose one out of then $n-1$ remaining objects), $\ldots$ and $n-r+1$ ways to fill the last position (to choose one out of then $n-r+1$ remaining objects).

Thus, according to the rule of product, we have
$P(n,r) = P(r,r) \times C(n,r).$
Using the notation

$$n! = n(n-1)(n-2)\ldots 3\times 2\times 1,\ for\ n>1,\ we\ get$$

$$P(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 3\times 2\times 1}{(n-1)(n-2)\ldots 3\times 2\times 1}$$
$$= \frac{n!}{(n-1)!}$$

□

Problem 5.1.8. In how many ways can n people stand to form a ring?

Solution. If we pick a particular person and let him occupy a fixed position, the remaining $n-1$ people will be arranged using this fixed position as reference in a ring. Again, there are $(n-1)!$ ways of arranging these $n-1$ people. ■

Result 5.1.9. Derive an expression for the number of ways of arranging n objects in which all of them are not distinct.

Proof. Let there be n objects that are not all distinct. Specifically, let there be q_1 objects of the first kind, q_2 objects of the second kind, $\ldots$ and q_t objects of the t^{th} kind.

Imagine that the n objects are marked so that objects of the same kind become distinguishable from one another. There are, of course, $n!$ ways in which these n "distinct" objects can be permuted. However, two permutations will be the same when the marks are erased if they differ only in the arrangement of marked objects that are of the same kind. Therefore, each permutation of the unmarked objects will correspond to $q_1!q_2!\ldots q_t!$ permutations of the marked objects. Then the number of $n-$ permutations of these n objects is given by

$$\frac{n!}{q_1!q_2!\ldots q_t!}$$

□

Problem 5.1.10. In how many ways can 5 dashes and 8 dots can be arranged?

Solution. Out of 13 symbols, 5 dashes are of one kina and 8 dots are of another kind and hence by previos result, the number of ways of different arrangements is

$$\frac{13!}{5!8!} = 1,287$$

■

Remark 5.1.11. If we use only seven of the thirteen dashes and dots, there are

$$\frac{7!}{5!2!} + \frac{7!}{4!3!} + \frac{7!}{3!4!} + \frac{7!}{2!5!} + \frac{7!}{1!6!} + \frac{7!}{7!} = 120$$

distinct representations.

Problem 5.1.12. Show that $(k!)!$ is divisible by $(k!)^{(k-1)!}$ for any integer k.

Solution. We consider a collection of $k!$ objects among which there are k of the first kind, k of the second kind,..., and k of the $(k-1)!$ th kind. The total number of ways of permuting these objects is given by

$$\frac{(k!)!}{(k!)!(k!)!\dots(k!)!} = \frac{(k!)!}{(k!)^{(k-1)!}}$$

Since the total number of permutations must be an integral value, $(k!)!$ is divisible by $(k!)^{(k-1)!}$ ■

Result 5.1.13. The number of ways to arrange r objects when they are selected out of n distinct objects with unlimited repetitions is n^r.

Proof. Since there are n ways to choose an object to fill the first position, n ways to choose an object to fill the second position,..., and n ways to choose an object to fill the r th position, by rule of product, the number of ways n^r. □

Problem 5.1.14. Among 10 billion numbers between 1 and $10,000,000,000,$ how many of them contain the digit 1 ? How many of them do not?

Solution. Among the 10 billion numbers between 0 and $9,999,999,999,$ there are 9^{10} numbers that do not contain the digit $1.$ Therefore, among the 10 billion numbers between 1 and $10,000,000,000,$ there are $9^{10}-1$ numbers that do not contain the digit 1 and $10^{10}-(9^{10}-1)$ numbers that contain digit 1. ■

Definition 5.1.15. A *binary* sequence is a sequence of $0'$ s and $1'$ s.

A *ternary* sequence is a sequence of $0'$ s, $1'$ s and $2'$ s.

A *quaternary* sequence is a sequence of $0'$ s, $1'$ s , $2'$ s and $3'$ s.

Problem 5.1.16. What is the number of n-digit binary sequences that contain an even number of 0 's (zero is considered as an even number)?

Solution. The problem is immediately solved if we observe that because of symmetry half of the 2^n n-digit binary sequences contain an even number of $0'$ s and the other half of the sequences contain an odd number of $0'$ s.

Another way to look at the problem is to consider the 2^{n-1} $(n-1)$ digit binary sequences. If an $(n-1)-$ digit binary sequence contains an even number of $0'$ s, we can append to it a 1 as the $n-$ th digit to yield an n-digit binary sequence that contains an even number of $0'$ s. If an $(n-1)-$ digit binary sequence contains an odd number of $0'$ s, we can append to it a 0 as the n-th digit to yield an n-digit binary sequence that contains an even number of $0'$ s. Therefore, there are 2^{n-1} n-digit binary sequences which contain an even number of $0'$ s. ∎

Remark 5.1.17. Consider the n-digit quaternary sequences. Again, because of symmetry, there are $\frac{4^n}{2}$ sequences in each of which the total number of $0'$ s and $1'$ s is even.

Problem 5.1.18. Find the number of quaternary sequences that contain an even number of 0 's.

Solution. We divide the 4^n sequences into two groups: the 2^n sequences that contain only $2'$ s and $3'$ s and the $4^n - 2^n$ sequences that contain one or more $0'$ s or $1'$ s. The sequences in the first group are, of course, sequences that have an even number of $0'$ s. The sequences in the second group can be subdivided into categories according to the patterns of $2'$ s and $3'$ s in the sequences. (For instance, sequences of the pattern $23xx2x3xxx$ will be in one category where the x' s are $0'$ s and $1'$ s.) Since half of the sequences in each category have an even number of $0'$ s, the total number of sequences that have an even number of $0'$ s in the second group is $(4^n - 2^n)/2$. Therefore, among the 4^n n-digit quaternary sequences, there are $2^n + (4^n - 2^n)/2$ sequences that have an even umber of $0'$ s. ∎

Combinations

According to the result $5.1.5,$ the number of r-combinations of n objects is

$$C(n,r) = \frac{P(n,r)}{r} = \frac{n!}{r!(n-r)!}$$

It is immediately obvious from this formula that

$$C(n,r) = C(n,n-r).$$

This indeed is what one would expect since selecting r objects out of n objects is equivalent to picking the $n-r$ objects that are not to be selected.

Problem 5.1.19. If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?

Solution. First of all, the number of diagonals is equal to $C(10,2) - 10 = 45 - 10 = 35$ as there are $C(10,2)$ straight lines joining the $C(10,2)$ pairs of vertices but 10 of these 45 lines are the sides of the decagon. Since for every four vertices we can count exactly one intersection between the diagonals as Figure $5.1.1$ shows (the decagon is convex), there is a total of $C(10,4) = 210$ intersections between the diagonals.

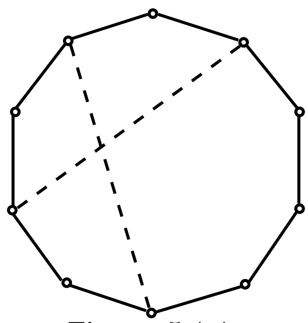


Figure $5.1.1$

Since a diagonal is divided into $k+1$ straight-line segments when there are k intersecting points lying along it and since each intersecting point lies along two diagonals, the total number of straight-line segments into which the diagonals are divided is $35 + 2 \times 210 = 455.$ ■

Problem 5.1.20. Eleven scientists are working on a secret project. They wish to lock up the documents in a cabinet such that the cabinet can be opened if and only if six or more of the scientists are present. What is the smallest number of locks needed? What is the smallest number of keys to the locks each scientist must carry?

Solution. (a) To answer the first question, observe that for any group of five scientists, there must be at least one lock they cannot open. Moreover, for any two different groups of five scientists, there must be two different locks they cannot open, because if both groups cannot open the same lock, there is a group of six scientists among these two groups who will not be able to open the cabinet. Thus, at least $C(11,5) = 462$ locks are needed.

(b) As to the number of keys each scientist must carry, let A be one of the scientists. Whenever A is associated with a group of five other scientists, A should have the key to the lock(s) that these five scientists were not able to open. Thus, A carries at least $C(10,5) = 252$ keys. ■

Problem 5.1.21. In how many ways can three numbers be selected from the numbers $1, 2, \ldots, 300$ such that their sum is divisible by 3?

Solution. The 300 numbers $1, 2, \ldots, 300$ can be divided into three groups: those that are divisible by $3,$ those that yield the remainder 1 when divided by $3,$ and those that yield the remainder 2 when divided by $3.$ Clearly, there are 100 numbers in each of these groups. If three numbers from the first group are selected, or if three numbers from the second group are selected, or if three numbers from the third group are selected, or if three numbers, one from each of the three groups, are selected, their sum will be divisible by $3.$ Thus, the total number of ways to select three desired numbers is

$$C(100, 3) + C(100, 3) + C(100, 3) + (100)^3 = 1, 485, 100$$ ■

Result 5.1.22. When repetitions in the selection of the objects are allowed, the number of ways of selecting r objects from n distinct objects is

$$C(n + r - 1, r)$$

Proof. Let the n objects be identified by the integers $1, 2, \ldots, n$ and let a specific selection of r objects be identified by a list of the corresponding integers $\{i, j, k, \ldots, m\}$ arranged in increasing order. For example, the selection in which the first object is selected thrice, the second object is not selected, the third object is selected once, the fourth object is selected once, the fifth object is selected twice, etc., is represented as $\{1, 1, 1, 3, 4, 5, 5, \ldots\}.$ To the r integers in such a list we add 0 to the first integer,1 to the second integer,. . . and $r - 1$ to the r th integer. Thus, $\{i, j, k, \ldots, m\}$ becomes $\{i, j + 1, k + 2, \ldots, m + (r - 1)\}.$
For example, the selection $\{1, 1, 1, 3, 4, 5, 5, \ldots\}$ becomes $\{1, 2, 3, 6, 8, 10, 11, \ldots\}.$ Since each selection will then be identified uniquely as a selection of r distinct integers from the integers $1, 2, \ldots, n + (r - 1),$ we get

$$C(n + r - 1, r).$$

□

Problem 5.1.23. Out of a large number of pennies, nickels, dimes, and quarters, in how many ways can six coins be selected?

Solution. The answer is $C(4 + 6 - 1, 6) = C(9, 6) = 84,$ because this is the same as selecting six coins from a penny, a nickle, a dime, and a quarter with unlimited repetitions.

Problem 5.1.24. What is the the number of outcomes when three distinct dice are rolled,

(a) when they are distinct

(b) when they are indistinguishable?

Solution. (a) When three distinct dice are rolled, and they are distinct. This can be seen by considering the selection of three numbers from the six numbers $1, 2, 3, 4, 5, 6$ when repetitions are allowed. Hence the number of outcomes is $6 \times 6 \times 6 = 216,$

(b) If the three dice are indistinguishable, the number of outcomes is $C(6+3-1, 3) = 56.$ ■

Result 5.1.25. When the objects are not all distinct, the number of ways to select one or more objects from them is equal to $(q_1+1)(q_2+1)\ldots(q_t+1)-1$
where there are q_1 objects of the first kind, q_2 objects of the second kind, $\ldots,$ and q_t objects of the t^{th} kind.

Proof. This result follows directly from the rule of product. There are q_1+1 ways of choosing the object of the first kind, i.e., choosing none of them, one of them, two of them, $\ldots,$ or q_1 of them. Similarly, there are q_2+1 ways of choosing objects of the second kind, $\ldots,$ and q_t+1 ways of choosing objects of the t^{th} kind. The term -1 corresponds to the "selection" in which no object at all is chosen and should be discounted.

□

Problem 5.1.26. How many divisors does the number 1400 have?

solution Since $1400 = 2^3 \times 5^2 \times 7,$ the number of its divisors is

$(3+1)(2+1)(1+1) = 24$

which is equal to the number of ways to select the prime factors of $1400.$ (Both 1 and 1400 are considered to be divisors of the number $1400.$) ■

Exercises

1. a. Use the relation $C(n,r) = C(n-1,r) + C(n-1,r-1)$ to prove the identity $C(n+1,m) = C(n,m)+C(n-1,m-1)+C(n-2,m-2)+\ldots+C(n-m,0)$ for $m \leq n.$

   b. Prove this identity using combinatorial arguments.

2. a. Prove the identity

   $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1.$

   b. Discuss the combinatorial significance of this identity.

   c. Show that any integer m can be expressed uniquely in the following form (factorial representation):

   $m = a_1 1! + a_2 2! + a_3 3! + \ldots + a_i i! + \ldots$ where $0 \leq a_i \leq i$ for $i = 1, 2, \ldots$

3. Give a combinatorial argument to prove that

   a. $P(n,n) = P(n,n-1)$

   b. $P(n,n) \neq P(n,n-2)$

4. Use a combinatorial argument to prove the identity

   a. $C(n,0)+C(n,1)+C(n,2)+\ldots+C(n,n)=2^n$

   b. $n \times C(n-1,r) = (r+1) \times C(n,r+1)$

5. Give a combinatorial argument to prove that $C(n,1)+2\times C(n,2)+3\times C(n,3)+\ldots+n\times C(n,n)=n\times 2^n-1$

6. a. Use a combinatorial argument to prove that $(2n)!/2^n$ and $(3n)!/(2^n \times 3^n)$ are integers.

   b. Prove that $(n^2)!/(n!)^{n+1}$ is an integer.

7. Three integers are selected from the integers $1, 2, \ldots 1000$. In how many ways can these integers be selected such that their sum is divisible by 3?

8. a. Among $2n$ objects, n of them are identical. Find the number of ways to select n objects out of these $2n$ objects.

   b. Among $3n+1$ objects, n of them are identical. Find the number of ways to select n objects out of these $3n+1$ objects.

9. From n distinct integers, two groups of integers are to be selected with k_1 integers in the first group and k_2 integers in the second group, where k_1 and k_2 are fixed and $k_1+k_2 \leq n$. In how many ways can the selection be made such that the smallest integer in the first group is larger than the largest integer in the second group?

10. Suppose that no three of the diagonals of a convex $n-$ gon meet at the same point inside of the $n-$ gon. Find the number of different triangles the sides of which are made up of the sides of the $n-$ gon, the diagonals, and segments of the diagonals.

11. Show that the number of $n-$ digit quaternary sequences that have an even number of $0'$ s and an even number of $1'$ s is $(4^n/4)+(2^n/2)$.

5.2 Distribution of Distinct and Non-distinct objects

In the previous section about the permutation of objects, we introduced the notion of placing distinct objects into distinct cells. Two cases must be considered. First, for $n \geq r,$ there are $P(n,r)$ ways to place r distinct objects into n distinct cells, where each cell can hold only one object. As was shown before, the first object can be placed in one of then cells, the second object can be placed in one of then $n-1$ remaining cells, etc. On the other hand, for $r \geq n,$ there are $P(r,n)$ ways to place n of r distinct

objects into n distinct cells, where each cell can hold only one object. The argument is similar to the one above; that is, there are r ways to select an object to be placed in the first cell, $r-1$ ways to select an object to be placed in the second cell, etc.
The distribution of r distinct objects in n distinct cells where each cell can hold any number of objects is equivalent to the arrangement of r of the n cells when repetitions are allowed. In terms of the distribution of distinct objects in distinct cells, since the first object can be placed in one of the n cells, the second object can again be placed in one of the n cells, etc., there are nr ways of distributing the objects.
Notice that in the above case, when more than one object is placed in the same cell, the objects are not ordered inside the cell. When the order of objects in a cell is also considered, the number of ways of distribution is

$$\frac{(n+r-1)!}{(n-1)!} = (n+r-1)(n+r-2)\dots(n+1)n$$

To prove this result, we imagine such a distribution as an ordered arrangement of the r (distinct) objects and the $n-1$ (nondistinct) intercell partitions. Using the previously derived formula for the permutation of $r+n-1$ objects where $n-1$ of them are of the same kind, we obtain the result $(n+r-1)!/(n-1)!$.

There is an alternative way to derive this formula. There are n ways to distribute the first object. After the first object is placed in a cell, it can be considered as an added partition that divides the cell into two cells. Therefore, there are $n+1$ ways to distribute the second object. Similarly, there are $n+2$ ways to distribute the third object, $\dots,$ and $n+r-1$ ways to distribute the r th object.

Example 5.2.1. Find the number of ways of arranging seven flags on five masts when all the flags must be displayed but not all the masts have to be used.
Solution If there is a single flag on a mast, we assume that it is raised to the top of the mast; however, if there is more than one flag on a mast, the order of the flags on the mast is important. Henc the total number of ways is $5\times 6\times 7\times 8\times 9\times 10\times 11$.

Similarly, seven cars can go through five toll booths in $5\times 6\times 7\times 8\times 9\times 10\times 11$ ways.

The distribution of n objects (q_1 of them are of one kind, q_2 of them are of another kind,..., and q_t of them are of the t^{th} kind) into n distinct cells (each of which can hold only one object) is equivalent to the permutation of these objects. Thus, the number of ways of distribution is

$$\frac{n!}{q_1!q_2!\dots q_t!}$$

Among then distinct cells, we have $C(n, q_1)$ ways to pick q_1 cells for the objects of the first kind, $C(n - q_1, q_2)$ ways to pick q_2 cells for the objects of the second kind, etc. The number of ways of distribution is, therefore,

$$C(n, q_1)C(n-q_1, q_2)C(n-q_1-q_2, q_3)\dots C(n-q_1-q_2\dots-q_{t-1}, q_t)$$

$$\begin{aligned} &P(n-q_1-q_2\dots-q_t, n-q_1-q_2\dots-q_t) \\ &\qquad = \frac{n!}{q_1!(n-q_1)!}\frac{(n-q_1)!}{q_2!(n-q_1-q_2)!}\frac{(n-q_1-q_2)!}{q_3!(n-q_1-q_2-q_3)} \\ &\qquad \dots\frac{(n-q_1-q_2\dots-q_{t-1})!}{q_t!(n-q_1-q_2\dots-q_t)!}(n-q_1-q_2\dots-q_t)! \end{aligned}$$

The factor $P(n-q_1-q_2\dots-q_t, n-q_1-q_2\dots-q_t)$ is the number of ways of permuting those objects that are one of a kind. It follows that the number of ways of distributing r objects $(r \leq n)$, with q_1 of them of one kind, q_2 of them of another kind, etc., into n distinct cells is

$$C(n, q_1)C(n-q_1, q_2)C(n-q_1-q_2, q_3)\dots C(n-q_1-q_2\dots-q_{t-1}, q_t)$$

$$\begin{aligned} &P(n-q_1-q_2\dots-q_t, r-q_1-q_2\dots-q_t) \\ &\qquad = \frac{n!}{q1_!q_2!\dots q_t!}\frac{1}{(n-r)!} \end{aligned}$$

Distribution of Non-distinct objects

In terms of the distribution of objects into cells, there are $C(n, r)$ ways of placing r nondistinct objects into n distinct cells with at most one object in each cell $(n \geq r)$; this follows because the distribution can be visualized as the selection of r cells from the n cells for the r nondistinct objects.

The number of ways to place r nondistinct objects into n distinct cells where a cell can hold more than one object is $C(n + r - 1, r)$. This result comes from the observation that distributing the r nondistinct objects is equivalent to selecting r of the n cells for the r objects with repeated selections of cells allowed. A different argument can be used to derive the result. Imagine the distribution of the r objects into n cells as an arrangement of the r objects and the $n - 1$ intercell partitions. Since both the objects and the partitions are nondistinct, the number of ways of arrangement is

$$\frac{(n-1+r)!}{(n-1)!r!} = C(n+r-1, r)$$

This result can be derived by using another argument. We can first select r cells from the n cells and then distribute the r objects into these r cells; that is,

$$C(n,r)\frac{r!}{q1_!q_2!\dots q_t!} = \frac{n!}{q1_!q_2!\dots q_t!}\frac{1}{(n-r)!}$$

If none of the n cells can be left empty (that means r must be larger than or equal to n), the number of ways of distribution is

$C(r-1, n-1)$

Since we can first distribute one object in each of the n cells and then distribute the remaining $r-n$ objects arbitrarily, the number of ways of distribution is

$$C((r-n)+n-1, r-n) = C(r-1, r-n) = C(r-1, n-1)$$

A direct extension of this result is the calculation of the number of ways of distributing r nondistinct objects into n distinct cells with each cell containing at least q objects. After placing q objects in each of the n cells, we have

$$C((r-nq)+n-1, r-nq) = C(n-nq+r-1, n-1)$$

Problem 5.2.2. Five distinct letters are to be transmitted through a communications channel. A total of 15 blanks are to be inserted between the letters with at least three blanks between every two letters. In how many ways can the letters and blanks be arranged?

Solution. There are $5!$ ways of arranging the letters. For each arrangement of the letters, we can consider the insertion of the blanks as placing 15 nondistinct objects into four distinct interletter positions with at least three objects in each interletter position. Therefore, the total number of ways of arranging the letters and blanks is $5! \times C(4-12+15-1, 4-1) = 5! \times C(6,3) = 2,400$ ∎

Problem 5.2.3. In how many ways can $2n+1$ seats in a congress be divided among three parties so that the coalition of any two parties will ensure them of a majority?

Solution. This is a problem of distributing $2n+1$ nondistinct objects into three distinct cells. Without any restriction on the number of seats each party can have, there are

$$C(3+(2n+1)-1, 2n+1) = C(2n+3, 2n+1) = C(2n+3, 2)$$

ways of distributing the seats. However, among these distributions, there are some in which a party gets $n+1$ or more seats. For a particular party to have $n+1$ or more seats, there are $C(3+n-1, n) = C(n+2, n) = C(n+2, 2)$ ways of distributing the seats. The ways of distribution are enumerated by giving the particular party $n+1$ seats first and then dividing the remaining n seats among the three parties arbitrarily. Therefore, the total number of ways to divide the seats so that no party alone will have a majority is

$$\begin{aligned}&C(2n+3,2)-3\times C(n+2,2)\\&=\frac{1}{2}(2n+3)(2n+2)-\frac{3}{2}(n+2)(n+1)=\frac{n}{2}(n+1)\end{aligned}$$

When there are $2n$ seats, the total number of ways of dividing the seats becomes $C(2n+2,2)-3\times C(n+2,2)+3=\frac{1}{2}(n-1)(n-2)$

The term $C(2n+2,2)$ is the total number of ways of distributing the $2n$ seats. Similarly, $C(n+2,2)$ is the number of ways of distributing the $2n$ seats such that a particular party gets n or more seats. The term $+3$ is due to the fact that each of the three distributions $(n,n,0)$, $(n,0,n)$, $(0,n,n)$ is accounted for twice in the term $3C(n+2,2)$. ■

Exercises

1. Five teaching machines are to be used by a group of m students. If the same number of students should be assigned to use the first and the second machines, in how many ways can the assignment be made?
2. Among the set of 10^n $n-$ digit integers, two integers are considered to be equivalent if one can be obtained by a permutation of the digits of the other.

   a. How many nonequivalent integers are there?

   b. If the digits 0 and 9 can appear at most once, how many nonequivalent integers are there, for $n \geq 2$?
3. In how many ways can the letters $a, a, a, a, a, b, c, d, e$ be permuted such that no two a' s are adjacent?
4. Consider the set of words of length n generated from the alphabet $\{0, 1, 2\}$.

   a. Show that the number of words in each of which the digit 0 appears an even number of times is $(3^n+1)/2$.

   b. Prove the identity

   $$\binom{n}{0}2^n+\binom{n}{2}2^{n-2}+\ldots+\binom{n}{q}2^{n-q}=\frac{3^n+1}{2}$$

   where $q=n$ when n is even, and $q=n-1$ when n is odd.

5. An alphabet of m letters can be transmitted through a communication channel. Find the number of different messages of n letters, if

   a. The letters can be used repeatedly in a message.

   b. l of the m letters can be used only as the first and the last letters in a message; the other letters can appear anywhere with unrestricted repetitions in a message.

   c. l of the m letters can be used only as the first and the last letters in a message; the other letters can appear anywhere, except the two ends, with unrestricted repetitions in a message.

5.3 Generating Functions

From three distinct objects $a, b,$ and $c,$ there are three ways to choose one object, namely, to choose either a or b or $c.$ Let us represent these possible choices symbolically as $a + b + c.$ Similarly, from these three objects, there are three ways to choose two objects, namely, to choose either a and $b,$ or b and $c,$ or c and $a,$ which can be represented symbolically as $ab + bc + ca.$ There is only one way to choose three objects, which can be represented symbolically as $abc.$ Examining the polynomial

$$(1 + ax)(1 + bx)(1 + cx) = 1 + (a + b + c)x + (ab + bc + ca)x^2 + (abc)x^3$$

we discover that all these possible ways of selection are exhibited as the coefficients of the powers of $x.$ In particular, the coefficient of x^i is the representation of the ways of selecting i objects from the three objects. This, of course, is not sheer coincidence. We have an interpretation of the polynomial according to the rule of sum and the rule of product. Symbolically, the factor $1 + ax$ means that for the object $a,$ the two ways of selection are “not to select a ” or “to select $a.$ ” The variable x is a formal variable and is used simply as an indicator. The coefficient of x^0 shows the ways no object is selected, and the coefficient of x^1 shows the ways one object is selected. Similar interpretation can be given to the factors $1+bx$ and $1+cx.$ Thus, the product $(1+ax)(1+bx)(1+cx)$ indicates that for the objects $a, b,$ and $c,$ the ways of selection are “to select or not to select a ” and “to select or not to select b ” and “to select or not to select c.” It is clear that the powers of x in the polynomial indicate the number of objects that are selected, and the corresponding coefficients show all the possible ways of selection. This example motivates the formal definition of the generating function of a sequence.

Definition 5.3.1. Let $(a_o, a_1, a_2, \ldots, a_r, \ldots)$ be the symbolic representation of a sequence of events, or let it simply be a sequence of numbers. The function $F(x) = a_o\mu_o(x) + a_1\mu_1(x) + a_2\mu_2(x) + \ldots + a_r\mu_r(x) + \ldots$ is called the *ordinary generating function* of the sequence $(a_o, a_1, a_2, \ldots, a_r, \ldots)$, where $\mu_o(x), \mu_1(x), \mu_2(x), \ldots, \mu_r(x), \ldots$ is a sequence of functions of x that are used as indicators.

The *indicator functions*, the, $\mu(x)'$ s, are usually chosen in such a way that no two distinct sequences will yield the same generating function. Clearly, the generating function of a sequence is just an alternative representation of the sequence. For example, using $1, cos\ x, cos\ 2x, \ldots, cos\ rx, \ldots$ as the indicator functions, we see that the ordinary generating function of the sequence $(l, w, w^2, \ldots, w^r, \ldots)$ is

$$F(x) = 1 + wcos\ x + w^2 cos\ 2x + \ldots + w^r cos\ rx + \ldots$$

On the other hand, using $1, 1 + x, 1 - x, 1 + x^2, 1 - x^2, \ldots, 1 + x^r, 1 - x^r, \ldots$ as the indicator functions, the ordinary generating function of the sequence $(3, 2, 6, 0, 0)$ is

$$3 + 2(1 + x) + 6(1 - x) = 11 - 4x$$

However, the sequences $(1, 3, 7, 0, 0)$ and $(1, 2, 6, 1, 1)$ will also yield the same ordinary generating function; that is,

$1 + 3(1 + x) + 7(1 - x) = 11 - 4x$ and

$1 + 2(1 + x) + 6(1 - x) + (1 + x^2) + (1 - x^2) = 11 - 4x.$

Hence, we see that the functions $1, 1 + x, 1 - x, 1 + x^2, 1 - x^2, \ldots$ should not be used as indicator functions. The most usual and useful form of $u_r(x)$ is x^r. In that case, for the sequence $(a_0, a_1, a_2, \ldots, a_r, \ldots)$, we have $F(x) = a_o + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots$. We shall limit our discussion to indicator functions of this form. From now on, when we talk about the generating functions of a sequence, we shall mean the generating function of the sequence with the powers of x as indicator functions. Notice that the sequence $(a_o, a_1, a_2, \ldots, a_r, \ldots)$ can be an infinite sequence, and $F(x)$ will then be an infinite series. However, because x is just a formal variable, there is no need to question whether the series converges.

Generating Functions for Combinations

We have seen that the polynomial $(1 + ax)(1 + bx)(1 + cx)$ is the ordinary generating function of the different ways to select the objects $a, b,$ and $c.$ Instead of the different ways of selection, we may only be interested in the number of ways of selection. By setting $a = b = c = 1,$ we have

$$(1+x)(1+x)(1+x) = (1+x)^3 = 1 + 3x + 3x^2 + x^3$$

Clearly, we see that there is one way to select no objects from the three objects, $C(3,0)$, three ways to select one object out of three, $C(3,1)$, etc. Usually, a generating function that gives the number of combinations or permutations is called an *enumerator*. In particular, an ordinary generating function that gives the number of combinations or permutations is called an *ordinary enumerator.*

This notion can be extended immediately. To find the number of combinations of n distinct objects, we have the ordinary enumerator

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots + x^n$$

$$= C(n,0) + C(n,1)x + C(n,2)x^2 + \ldots + C(n,r)x^r + \ldots + C(n,n)x^n$$

An alternative point of view can also be taken. Except for the case $a_0 = \infty$, $F(x)$ converges at $x = 0$. Therefore, with the understanding that the value of x is set to be 0, we can carry the expression for $F(x)$ along in our computation without concerning ourselves further with the convergence problem.

In the expansion of $(1+x)^n$, the coefficient of the term x^r is the number of ways the term x^r can be formed by taking $r\ x'$ s and $n - r\ 1'$ s among the n factors $1 + x$. It is for this reason that the $C(n,r)'$ s are called the *binomial coefficients.* In a binomial expansion, $\binom{n}{r}$ is a common alternative notation for $C(n,r)$.

Example 5.3.2. From

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n = (1+x)^n$$

we have the identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n} = 2^n$$

by setting x equal to 1. The combinatorial significance of this identity is that both sides give the number of ways of selecting none, or one, or two,…, or n objects out of n distinct objects. We also have the identity

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^r\binom{n}{r} + \ldots + (-1)^n\binom{n}{n} = 0$$

by setting x equal to -1. Writing this as

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots$$

we see that the number of ways of selecting an even number of objects is equal to the number of ways of selecting an odd number of objects from n distinct objects.

Problem 5.3.3. Prove the identity

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{r}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$$

Solution. *Method 1* We observe that the expression on the left-hand side is the constant term in $(1+x)^n(1+x^{-1})^n$. Since

$$(1+x)^n(1+x^{-1})^n=(1+x)^n(1+x)^nx^{-n}=x^{-n}(1+x)^{2n}$$

and the constant term in $x^{-n}(1+x)^{2n}$ is $\binom{2n}{n}$, we have proved the identity.
Method 2 We rewrite the identity to be proved as

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\ldots+\binom{n}{r}\binom{n}{n-r}$$

$$+\ldots+\binom{n}{n}\binom{n}{0}=\binom{2n}{n}$$

and use a combinatorial argument. To select n objects out of $2n$ objects, we shall first divide them (in any arbitrary manner) into two piles with n objects in each pile. There are $\binom{n}{i}$ ways to select i objects from the first pile and $\binom{n}{n-i}$ ways to select $n-i$ objects from the second pile to make up a selection of n objects. Therefore, the number of ways to make the selection is $\sum\limits_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$ which is also equal to $\binom{2n}{n}$.
To see an application of this result,let us consider the problem of finding the number of $2n-$ digit binary sequences which are such that the number of $0'$ s in the first n digits of a sequence is equal to the number of $0'$ s in the last n digits of the sequence. Since the number of $n-$ digit binary sequences containing $r\ 0'$ s is $\binom{n}{r}$, the number of $2n-$ digit binary sequences containing $r\ 0'$ s in the first n digits as well as in the last n digits is $\binom{n}{r}^2$. Therefore, the number of $2n-$ digit binary sequences which are such that the number of $0'$ s in the first n digits of a sequence is equal to the number of $0'$ s in the last

n digits of the sequence is

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{r}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$$

■

Problem 5.3.4. Prove the identity

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + r\binom{n}{r} + \ldots + n\binom{n}{n} = n2^{n-1}$$

Solution. Differentiating both sides of the identity

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n = (1+x)^n$$

we have

$$\binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{2}x^2 \ldots + r\binom{n}{r}x^{r-1} + \ldots + n\binom{n}{n}x^{n-1} = n(1+x)^{n-1}$$

The given identity is obtained by setting x equal to $1.$ ■

Problem 5.3.5. What is the coefficient of the term x^{23} in $(1+x^5+x^9)^{100}$?

Solution. Since $x^5x^9x^9 = x^{23}$ is the only way the term x^{23} can be made up in the expansion of $(1+x^5+x^9)^{100}$ and there are $C(100,2)$ ways to choose the two factors x^9 and then $C(98,1)$ ways to choose the factor x^5, the coefficient of x^{23} is $C(100,2) \times C(98,1) = \frac{100\times 99}{2} \times 98 = 485,100.$ ■

Problem 5.3.6. Show that the ordinary generating function of the sequence

$$\binom{0}{0}, \binom{2}{1}, \binom{4}{2}, \ldots, \binom{2r}{r}, \ldots \text{ is } (1-4x)^{\frac{-1}{2}}.$$

Solution. According to the binomial theorem, we have

$$(1-4x)^{\frac{-1}{2}} = 1 + \sum_{r=1}^{\infty} \frac{(-1/2)(-1/2-1)\ldots(-1/2-r+1)(-4x)^r}{r!}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{4^r(1/2)(3/2)(5/2)\ldots[(2r-1)/2]}{r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{2^r (1.3.5. \ldots .(2r-1))}{r!} x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{2^r r! (1.3.5. \ldots .(2r-1))}{r! r!} x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(2.4.6 \ldots 2r)(1.3.5. \ldots (2r-1))}{r! r!} x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(2r)!}{r! r!} x^r$$

$$= 1 + \sum_{r=1}^{\infty} \binom{2r}{r} x^r$$

■

Problem 5.3.7. Evaluate the $\sum_{i=0}^{t} \binom{2i}{i}\binom{2t-2i}{t-i}$ for a given t.

Solution. Since $\binom{2i}{i}$ is the coefficient of the term x^i in $(1-4x)^{\frac{-1}{2}}$ and $\binom{2t-2i}{t-i}$ is the coefficient of the term x^{t-i} in $(1-4x)^{\frac{-1}{2}}$, $\sum_{i=0}^{t} \binom{2i}{i}\binom{2t-2i}{t-i}$ is the coefficient of the term x^t in $(1-4x)^{\frac{-1}{2}}(1-4x)^{\frac{-1}{2}} = (1-4x)^{-1}$

$$= 1 + 4x + (4x)^2 + (4x)^3 + \ldots + (4x)^r + \ldots,$$

we have,

$$\sum_{i=0}^{t} \binom{2i}{i}\binom{2t-2i}{t-i} = 4^t.$$

■

Selection with repetitions

When repetitions are allowed in the selections (or equivalently, when there is more than one object of the same kind), the extension is immediate. For example, the polynomial

$$(1 + ax + a^2x^2)(1 + bx)(1 + cx) = 1 + (a + b + c)x + (ab + bc + ac + a^2)x^2$$

$$+(abc + a^2b + a^2c)x^3 + (a^2bc)x^4.$$

is the ordinary generating function for the combinations of the objects $a, b,$ and $e,$ where a can be selected twice. Notice the difference between the combinatorial significance of this polynomial and that of the polynomial $(1 + ax)(1 + a^2x^2)(1 + bx)(1 + cx),$ which

can be written as $(1 + ax + a^2x^2 + a^3x^3)(1 + bx)(1 + cx)$.
As another example, let us consider the generating function

$$(1 + ax)(1 + a^2x)(1 + bx)(1 + cx) = 1 + (a + b + c + a^2)x$$

$$+(ab + bc + ac + a^3 + a^2b + a^2c)x^2 + (abc + a^3b + a^2bc + a^3c)x^3 + (a^3bc)x^4$$

We can imagine that there are four boxes, one containing $a,$ one containing two a' s, one containing $b,$ and one containing $c.$ The generating function gives the outcomes of the selection of the boxes.
Similarly, the ordinary enumerator for the combinations of the objects $a, b,$ and $c,$ where a can be selected twice, is

$$(1 + x + x^2)(1 + x)^2 = 1 + 3x + 4x^2 + 3x^3 + x^4.$$

The significance of the factor $1 + x + x^2$ is that for the object $a,$ there is one way not to select it, one way to select it once, and also one way to select it twice.

Example 5.3.8. Given two each of p kinds of objects and one each of q additional kinds of objects, in how many ways can r objects be selected?

Solution. The ordinary enumerator for the combinations is
$(1 + x + x^2)^p(1 + x)^q$
The coefficient of x^r in the enumerator is

$$\sum_{i=0}^{[\frac{r}{2}]} \binom{p}{i}\binom{p+q-i}{r-2i}$$

where $[r/2]$ denotes the integral part of $r/2$ (that is, $[r/2] = r/2 \; if \; r$ is even, and $[r/2] = (r-1)/2 \; if \; r$ is odd), because among the p factors of the form $(1 + x + x^2),$ we can select i x^2 's, and among the $p - i$ remaining factors of the form $(1 + x + x^2)$ and the q factors of the form $1 + x$ we can select $r - 2i$ x 's. ■

Example 5.3.9. The ordinary enumerator for the selection of r objects out of n objects with unlimited repetitions is $(1 + x + x^2 + \ldots + x^k + \ldots)^n$

$$= (\frac{1}{1-x})^n$$

$$= (1 - x)^{-n}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)\dots(-n-r+1)}{r!}(-x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(n)(n+1)\dots(n+r-1)}{r!}x^r$$

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

■

Example 5.3.10. The ordinary enumerator for the selection of r objects out of n objects $(r \geq n)$, with unlimited repetitions but with each object included in each selection, is $(x + x^2 + \dots + x^k + \dots)^n = x^n(\frac{1}{1-x})^n$

$$= x^n(1-x)^{-n}$$

$$= x^n \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

$$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^{n+i}$$

$$= \sum_{r=n}^{\infty} \binom{r-1}{r-n} x^r (let \text{ r=n+i})$$

■

Problem 5.3.11. Show that the number of ways in which r nondistinct objects can be distributed into n distinct cells, with the condition that no cell contains less than q nor more than $q+z-1$ objects, is the coefficient of x^{r-qn} in the expansion of $[(1-x^z)/(1-x)]^n$.

Solution. Since the ordinary enumerator for the ways a particular cell can be filled is

$$x^q + x^{q+1} + \dots + x^{q+z-1},$$

the ordinary enumerator for the distributions is

$$(x^q + x^{q+1} + \dots + x^{q+z-1})^n = x^{qn}(1 + x + \dots + x^{z-1})^n$$

$$= x^{qn}(\frac{1-xz}{1-x})^n$$

■

Problem 5.3.12. Find the number of ways in which four persons, are rolling a single die once, can have a total score of 17.

Solution. Taking $r = 17$, $n = 4$, $q = 1$, and $z = 6$, the ordinary enumerator is $x^4[(1-x^6)/(1-x)]^4$.
Since, $(1-x^6)^4 = 1 - 4x^6 + 6x^{12} - 4x^{18} + x^{24}$

$$(1-x)^{-4} = 1 + \frac{4}{1!}x + \frac{4\times 5}{2!}x^2 + \frac{4\times 5\times 6}{3!}x^3 + \ldots$$

The coefficient of x^{13} in $(1-x^6)^4(1-x)^{-4}$ is

$$\frac{4\times 5\times 6\times \ldots \times 16}{13!} - 4\frac{4\times 5\times 6\times \ldots \times 10}{7!} + 6\frac{4}{1!}$$

$$= \frac{14\times 15\times 16}{3!} - 4\frac{8\times 9\times 10}{3!} + 6\frac{4}{1!} = 104$$

■

Enumerators for Permutations

It is natural now for us to turn to the generating functions for permutations. However, there is an obvious difficulty when we try to extend our previous results. Since multiplication in the ordinary algebra in the field of real numbers (with which we are so familiar) is commutative (that is, $ab = ba$), we cannot quite handle the case of permutations using ordinary algebra.
The situation can be illustrated by an example of the permutations of the two objects a and b. What we want to have as a generating function for the permutations is

$$1 + (a+b)x + (ab+ba)x^2$$

However, this polynomial is equivalent to

$$1 + (a+b)x + (2ab)x^2$$

in which the two distinct permutations ab and ba can no longer be recognized. Instead of introducing a new algebra that is noncommutative for the case of permutations, we shall limit ourselves to the discussion of the enumerators for permutations which can still be handled by the ordinary algebra in the field of real numbers.

A direct extension of the notion of the enumerators for combinations indicates that an enumerator for the permutations of n distinct objects would have the form

$$\begin{aligned}F(x) &= P(n,0)x^0 + P(n,1)x + P(n,2)x^2 + P(n,3)x^3 + \ldots \\ &\quad + P(n,r)x^r + \ldots + P(n,n)x^n \\ &= 1 + \frac{n!}{(n-1)!}x + \frac{n!}{(n-2)!}x^2 + \frac{n!}{(n-3)!}x^3 + \ldots \\ &\quad + \frac{n!}{(n-r)!}x^r + \ldots + n!x^n\end{aligned}$$

Unfortunately, there is no simple closed-form expression for $F(x)$, and to carry along the polynomial in our manipulations certainly defeats the purpose of using the generating function representation. However, when we recall the binomial expansion

$$\begin{aligned}(1+x)^n &= 1 + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \ldots \\ &\quad + C(n,r)x^r + \ldots + C(n,n)x^r \\ &\quad = 1 + \frac{P(n,1)}{1!}x + \frac{P(n,2)}{2!}x^2 + \frac{P(n,3)}{3!}x^3 + \ldots \\ &\quad + \frac{P(n,r)}{r!}x^r + \ldots + \frac{P(n,n)}{n!}x^n\end{aligned}$$

we see the key to defining another kind of generating function, the *exponential generating function.*

Definition 5.3.13. Let $(a_0, a_1, a_2, \ldots, a_r, \ldots)$ be the symbolic representations of a sequence of events or simply be a sequence of numbers. Then the function

$$F(x) = \frac{a_0}{0!}\mu_0(x) + \frac{a_1}{1!}\mu_1(x) + \frac{a_2}{2!}\mu_2(x) + \ldots + \frac{a_r}{r!}\mu_r(x) + \ldots$$

is called the *exponential generating function* of the sequence $(a_0, a_1, a_2, \ldots, a_r, \ldots)$ with $\mu_0(x), \mu_1(x), \mu_2(x), \ldots, \mu_r(x), \ldots$ as the *indicator functions.*

Thus, $(1+x)^n$ is the exponential generating function of the $P(n,r)$'s with the powers of x as the indicator functions.

Definition 5.3.14. An exponential generating function that gives the number of combinations or permutations is called an *exponential enumerator.*

Example 5.3.15. We know that,
$(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + \ldots + C(n,r)x^r + \ldots + C(n,n)x^n$

Using the relation $P(n,r) = C(n,r) \times r!,$ we get

$$= 1 + \frac{P(n,1)}{1!} + \frac{P(n,2)}{2!}x^2 + \frac{P(n,3)}{3!}x^3 + \ldots + \frac{P(n,r)}{r!}x^r$$

Thus $(1+x)^n$ is the exponential generating function of the $P(n,r)$'s with the powers of x as the indicator functions.

Further Examples

1. The exponential generating function of the sequence $(1, 1 \times 3, 1 \times 3 \times 5, \ldots, 1 \times 3 \times 5 \times \ldots \times (2r+1)), \times$ is $(1-2x)^{-3/2}$
2. The exponential generating function of the sequence $(1, 1, 1 \ldots, 1, \ldots)$ is e^x.

Clearly, the exponential enumerator for the permutations of a single object with no repetitions is $1 + x$. We also see in the above that the exponential enumerator for the permutations of n distinct objects with no repetitions is $(1+x)^n$. (The definition of the exponential enumerator is actually chosen in such a way that the result will come out correctly.)

When repetitions are allowed in the permutations, the extension is immediate. The exponential enumerator for the permutations of all p of p identical objects is $x^P/p!$ since there is only one way of doing so. Thus, the exponential enumerator for the permutations of none, one, two,..., p of p identical objects is

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{p!}x^P$$

Similarly, the exponential enumerator for the permutation of all $p+q$ of $p+q$ objects, with p of them of one kind and q of them of another kind, is

$$\frac{x^P}{p!}\frac{x^q}{q!} = \frac{x^{p+q}}{p!q!}$$

which agrees with the known result that the number of permutations is $\frac{(p+q)!}{p!q!}$. It follows that the exponential enumerator for the permutations of none, one, two,..., $p+q$ of $p+q$ objects, with p of them of one kind and q of them of another kind, is

$$(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{p!}x^p)(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{q!}x^q)$$

For instance, the exponential enumerator for the permutations of two objects of one kind and three objects of another kind is

$$(1 + \frac{x}{1!} + \frac{x^2}{2!}) + (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}) =$$

$$1+(\frac{1}{1!}+\frac{1}{1!})x+(\frac{1}{1!1!}+\frac{1}{2!}+\frac{1}{2!})x^2+(\frac{1}{1!2!}+\frac{1}{1!2!}+\frac{1}{3!})x^3+(\frac{1}{1!3!}+\frac{1}{2!2!})x^4+(\frac{1}{2!3!})x^5$$

■

Result 5.3.16. The exponential enumerator for the number of $r-$ permutations of n distinct objects with unlimited repetitions is given e^{nx}.

Proof. The number of $r-$ permutations of n distinct objects with unlimited repetitions is given by the exponential enumerator

$$(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)^n = e^{nx} = \sum_{r=0}^{\infty}\frac{n^r}{r!}x^r.$$

■

□

Problem 5.3.17. Find the number of $r-$ digit quaternary sequences in which each of the digits $1, 2,$ and 3 appears at least once.

Solution. This problem is the same as that of permuting four distinct objects with the restriction that three of the four objects must be included in the permutations. The exponential enumerator for the permutations of the digit 0 is

$$(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots) = e^x$$

The exponential enumerator for the permutations of the digit $1(or\ 2,\ or\ 3)$ is

$$(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots) = e^x-1$$

It follows that the exponential enumerator for the permutations of the four digits is

$$\begin{aligned} e^x(e^x-1)(e^x-1)(e^x-1) &= e^x(e^{3x}-3e^{2x}+3e^x-1) \\ &= e^{4x}-3e^{3x}+3e^2x-e^x \\ &= \sum_{r=0}^{\infty}\frac{(4^r-3\times 3^r+3\times 2^r-1)}{r!}x^r \end{aligned}$$

Therefore, the number of $r-$ digit quaternary sequences in which each of the digits $1, 2,$ and 3 appears at least once $4^r-3\times 3^r+3\times 2^r-1.$ ■

Problem 5.3.18. Find the number of $r-$ digit quaternary sequences that contain an even number of $0'$ s.

Solution. The exponential enumerator for the permutations of the digit 0 is

$$(1 + x + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots) = \frac{1}{2}(e^x + e^{-x})$$

The exponential enumerator for the permutations of each of the digits $1, 2,$ and 3 is

$$(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) = e^x$$

It follows that the exponential enumerator for the number of quaternary sequences containing an even number of 0 's is

$$\begin{aligned}\frac{1}{2}(e^x + e^{-x})e^x e^x e^x &= \frac{1}{2}(e^{4x} + e^{2x})\\ &= 1 + \sum_{r=1}^{\infty} \frac{1}{2}\frac{(4^r + 2^r)}{r!}x^r\end{aligned}$$

Therefore, the number of $r-$ digit quaternary sequences that contain an even number of 0 's is

$$(4^r + 2^r)/2.$$

Similarly, to find the number of $r-$ digit quaternary sequences that contain an even number of 0 's and an even number of 1 's, we have the exponential enumerator

$$\begin{aligned}\frac{1}{2}(e^x + e^{-x})\frac{1}{2}(e^x + e^{-x})e^x e^x &= \frac{1}{4}(e^{2x} + 2 + e^{-2x})e^{2x}\\ &= \frac{1}{4}(e^{4x} + 2e^{2x} + 1)\\ &= 1 + \sum_{r=1}^{\infty} \frac{1}{2}\frac{(4^r + 2 \times 2^r)}{r!}x^r\end{aligned}$$

■

Example 5.3.19. Find the exponential enumerator for the number of ways to choose r or less objects from r distinct objects and distribute them into n distinct cells, with objects in the same cell ordered.

Solution. Notice that there are $C(r, m)$ ways to select m objects out of r objects and $n(n + 1) \ldots (n + m - 1)$ ways to arrange them in the n distinct cells. Since the value of m ranges from 0 to $r,$ the total number of ways is

$$C(r,0)+C(r,1)\times n+C(r,2)\times n(n+1)+C(r,3)\times n(n+1)(n+2)$$

$$+\ldots+C(r,r)\times n(n+1)\ldots(n+r-1)=r![\frac{1}{r!}\times 1+\frac{1}{(r-1)!1!}$$

$$\times n+\frac{1}{(r-2)!2!}\times n(n+1)+\frac{1}{(r-3)!3!}\times n(n+1)(n+2)$$

$$+\ldots\frac{1}{r!}n(n+1)\ldots(n+r-1)]$$

The expression in the square brackets is the coefficient of the term x^r in the product of the two series

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^r}{r!}+\ldots$$

and

$$(1-x)^{-n}=1+\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2+\ldots+\frac{n(n+1)(n+r-1)}{r!}x^r+\ldots$$

Therefore, $e^x/(1-x)^n$ is the exponential enumerator for the distributions of r or less objects into n distinct cells, with objects in the same cell ordered. ∎

Exercises

1. Among the three representatives from each of the 50 states, either none, or one, or two of them will be selected to form a special committee.

   a. In how many ways can the selection be made?

   b. If the committee has exactly 50 members, in how many ways can the selection be made?
   (The answer may be expressed as a summation.)

2. Find the value of a_{50} in the following expansion:

$$\frac{x-3}{x^2-3x+2}=a_0+a_1x+a_2x^2+\ldots+a_{50}x^{50}+\ldots$$

3. In how many ways can 200 identical chairs be divided among four conference rooms such that each room will have 20 or 40 or 60 or 80 or 100 chairs?

4. In how many ways can $3n$ letters be selected from $2n\ A's, 2n\ B's,$ and $2n\ C's?$

5. In how many ways can n letters be selected from an unlimited supply of $A's, B's,$ and $C's$ if each selection must include an even number of $A's$?
6. a. Let a_r denote the number of ways in which the sum r will show when two distinct dice are rolled. Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$.

   b. Let a_r denote the number of ways in which the sum r can be obtained by rolling a die any number of times. Show that the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$ is $(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$.
7. a. Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$ where a_r is the number of ways of selecting r objects from a set of six distinct objects, where each object can be selected not more than thrice.

   b. From the generating function found in part (a), determine the number of outcomes when three indistinguishable dice are rolled.
8. Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$ where a_r is the number of partitions of the integer r into distinct primes.
9. Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$ where a_r is the number of ways in which r letters can be selected from the alphabet $\{0, 1, 2\}$ with unlimited repetitions except that the letter 0 must be selected an even number of times.
10. Find the exponential generating function of the sequence $(1, 1\times 4, 1\times 4\times 7, \ldots, 1\times 4\times 7\times \ldots \times (3r+1), \ldots)$.
11. Let a_r denote the number of ways of permuting r of the 10 letters $A, A, A, A, B, C, C, D, E, E$. Find the exponential generating function of the sequence $(a_0, a_1, a_2, \ldots)$
12. Find the number of $n-$ digit words generated from the alphabet $\{0, 1, 2, 3, 4\}$ in each of which the total number of 0 's and l 's is even.
13. Find the number of $n-$ digit words generated from the alphabet $\{0, 1, 2\}$ in each of which none of the digits appears exactly three times.

5.4 Distribution of Distinct Objects into Nondistinct Cells

As examples on the use of exponential generating functions, we shall derive some results on the distribution of distinct objects into nondistinct cells. First we shall derive the number of ways of distributing r distinct objects into n distinct cells so that no cell is empty and the order of objects within a cell is not important. This problem can be viewed as finding the number of the $r-$ permutations of the n distinct cells with each cell included at least once in a permutation. The exponential enumerator for the permutations is

$$\begin{aligned}(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)^n=(e^x-1)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^i e^{(n-i)x} \\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^i\sum_{r=0}^{\infty}\frac{1}{r!}(n-i)^r x^r \\ &= \sum_{r=0}^{\infty}\frac{x^r}{r!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r\end{aligned}$$

Thus, the number of ways of placing r distinct objects into n distinct cells with no cell

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r=n!S(r,n)$$

where $S(r,n)$ is defined as $\frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$
and is called the *Stirling number of the second kind*.

Stirling numbers of the second kind, S(r,n)

r	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

The above table shows some of the Stirling numbers of second kind. It follows that the number of ways of placing r distinct objects into n nondistinct cells with no cell left empty is equal to $S(r,n)$. Previously we proved that there are n^r ways of placing r distinct objects into n distinct cells, when empty cells are allowed. When the cells become nondistinct, the number of ways is not equal to $n^r/n!$. As a matter of fact, the number of ways of distributing r distinct objects into n nondistinct cells with empty cells allowed is

$$S(r,1)+S(r,2)+\ldots+S(r,n) \quad for\; r\geq n$$

and is

$$S(r,1)+S(r,2)+\ldots+S(r,r) \quad for\; r\geq n \ldots\ldots (1)$$

These come directly from the argument that the number of ways of distributing r distinct objects into n nondistinct cells with empty cells allowed is equal to the number of ways of distributing these r objects so that one cell is not empty, or two cells are not empty, etc.

For the case of $r\leq n$ (i.e., there are at least as many cells as objects), there is a closed-form expression for the ordinary generating function of the numbers of ways of distributing the objects. Since $S(i,j)=0$ for $i<j,$ the count in the expression in (1) does not change if we add to it an infinite number of terms as follows:

$$S(r,1)+S(r,2)+\ldots+S(r,r)+S(r,r+1)+S(r,r+2)+\ldots\ldots (2)$$

Observe that

$$\frac{e^x-1}{1!}=S(0,1)+\frac{S(1,1)}{1!}x+\frac{S(2,1)}{2!}x^2+\ldots+\frac{S(r,1)}{r!}x^r+\ldots$$

$$\frac{(e^x-1)^2}{2!}=S(0,2)+\frac{S(1,2)}{1!}x+\frac{S(2,2)}{2!}x^2+\ldots+\frac{S(r,2)}{r!}x^r\ldots$$

$$\frac{(e^x-1)^k}{k!}=S(0,k)+\frac{S(1,k)}{1!}x+\frac{S(2,k)}{2!}x^2+\ldots+\frac{S(r,k)}{r!}x^r\ldots$$

$$\frac{(e^x-1)^r}{r!}=S(0,r)+\frac{S(1,r)}{1!}x+\frac{S(2,r)}{2!}x^2+\ldots+\frac{S(r,r)}{r!}x^r\ldots$$

$$\frac{(e^x-1)^{r+1}}{(r+1)!}=S(0,r+1)+\frac{S(1,r+1)}{1!}x+\frac{S(2,r+1)}{2!}x^2+\ldots+\frac{S(r,r+1)}{r!}x^r\ldots$$

Therefore, the coefficient of $x^r/r!,$ which is the number of ways of distributing r distinct objects into r or more nondistinct cells, in

$$\frac{e^x-1}{1!}+\frac{(e^x-1)^2}{2!}+\ldots+\frac{(e^x-1)^k}{k!}+\ldots+\frac{(e^x-1)^r}{r!}+\frac{(e^x-1)^{r+1}}{(r+1)!}\ldots\ldots(3)$$

is equal to the expression in (2). However, the generating function in (3) can be written as

$$e^{e^x-1}-1$$

Partition of Integers

As another illustration of the use of generating functions, we shall discuss the distribution of nondistinct objects into nondistinct cells.

Definition 5.4.1. A *partition of an integer* is a division of the integer into positive integral parts, in which the order of these parts is not important.

For example, $4, 3+1, 2+2, 2+1+1,$ and $1+1+1+1$ are the five different partitions of the integer $4.$

It is clear that a partition of the integer n is equivalent to a way of distributing n nondistinct objects into n nondistinct cells with empty cells allowed. We shall conduct our discussion in the context of the partitions of integers mainly because it is also an important topic in number theory. Observe that in the polynomial $1+x+x^2+x^3+x^4+\ldots+x^n,$ the coefficient of x^k is the number of ways of having k 1 's in a partition of the integer $n.$

Clearly, there is one way for $0 \leq k \leq n$ and no way for $k > n$ because in a partition of n there can be from no 1 's to at most n 1 's. It follows that in the infinite sum

$$1+x+x^2+x^3+x^4+\ldots+x^r+\ldots=\frac{1}{1-x},$$

the coefficient of x^k is the number of ways of having k 1 's in a partition of any integer larger than or equal to $k.$ Similarly, in the polynomial

$$1+x^2+x^4+x^6+x^8+\ldots+x^{[\frac{n}{2}]}$$

the coefficient of x^2k is the number of ways of having k 2 's in a partition of the integer $n.$ Also, in the infinite sum

$$1+x^2+x^4+x^6+x^8+\ldots+x^{2r}+\ldots=\frac{1}{1-x^2}$$

the coefficient of x^{2k} is the number of ways of having k 2 's in a partition of any integer larger than or equal to $2k$. Notice that a 2 in a partition will be accounted for by the term x^2, two 2's in a partition will be accounted for by the term x^4, etc. It follows then that

$$
\begin{aligned}
F(x) = {} & (1 + x + x^2 + x^3 + \ldots + x^r + \ldots) \\
& (1 + x^2 + x^4 + x^6 + \ldots + x^{2r} + \ldots) \\
& (1 + x^3 + x^6 + x^9 + \ldots + x^{3r} + \ldots) \\
& (1 + x^4 + x^8 + x^{12} + \ldots + x^{4r} + \ldots) \\
& \qquad \ldots (1 + x^n + x^{2n} + x^{3n} + \ldots + x^{nr} + \ldots) \\
= {} & \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots(1-x^n)}
\end{aligned}
$$

is the ordinary generating function of the sequence $(p(0), p(1), \ldots, p(n))$, where $p(i)$ denotes the number of partitions of the integer i. However, notice that $F(x)$ does not enumerate the $p(j)$'s for $j > n$; rather, it enumerates the number of partitions of the integer j that have no part exceeding n. For example, from

$$
\begin{aligned}
& \frac{1}{(1-x)(1-x^2)(1-x^3)} \\
& \qquad = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + \ldots
\end{aligned}
$$

we observe that there are three ways to partition the integer 3 and there are seven ways to partition the integer 6 such that the parts do not exceed 3. The ordinary generating function of the infinite sequence $(p(0), p(1), \ldots, p(n), \ldots)$, is

$$
F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}
$$

Remark 5.4.2. It is immediately clear that in

$$
\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots(1-x^{2n+1})},
$$

the coefficient of x^k for $k \le 2n+1$ is the number of partitions of the integer k into odd parts, and the coefficient of x^k for $k > 2n+1$ is the number of partitions of the integer k into odd parts not exceeding $2n+1$.
Similarly, in

$$
\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots},
$$

the coefficient of x^k is the number of partitions of the integer k into odd parts.

Remark 5.4.3. Also in

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\dots(1-x^{2n})},$$

and the coefficient of x^k for $k \leq 2n$ is the number of partitions of the integer k into even parts, and the coefficient of x^k for $k > 2n$ is the number of partitions of the integer k into even parts not exceeding $2n.$ Again, in

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\dots},$$

the coefficient of x^k is the number of partitions of the integer k into even parts.

Remark 5.4.4. Also, the polynomial

$$(1+x)(1+x^2)(1+x^3)\dots(1+x^n)$$

enumerates the partitions of integers no larger than n into distinct(unequal) parts and the partitions of integers larger than n into distinct parts not exceeding $n,$ and

$$(1+x)(1+x^2)(1+x^3)\dots(1+x^n)\dots$$

enumerates the partitions of the integers into distinct parts.

Problem 5.4.5. Prove that the number of partitions of an integer into distinct parts is equal to the number of partitions of the integer into odd parts.

Solution. Since $(1+x)(1+x^2)(1+x^3)\dots(1+x^r)\dots$

$$\begin{aligned}
&= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\dots\frac{1-x^{2r}}{1-x^r}\dots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)}\dots
\end{aligned}$$

we conclude that the number of partitions of an integer into distinct parts is equal to the number of partitions of the integer into odd parts.
For instance, the integer 6 can be partitioned into distinct parts in four different ways, namely,

$$6, 5+1, 4+2, 3+2+1$$

There are also exactly four different ways in which 6 can be partitioned into odd parts. They are $5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1.$ ■

Problem 5.4.6. Prove that any integer can be expressed as the sum of a selection of integers $1, 2, 4, 8, \ldots, 2^r, \ldots$ (without repetition) in exactly one way. (This is the well-known fact htat a decimal number cn be represented uniquely as a binary number).

Solution. Since

$$\begin{aligned}
&(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2r})\ldots \\
&\qquad = (1-x^2)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2r})\ldots \\
&\qquad = (1-x^4)(1+x^4)(1+x^8)\ldots(1+x^{2r})\ldots \\
&\qquad = 1
\end{aligned}$$

We have the identity

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2r})\ldots$$

Recalling that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

we conclude that any integer can be expressed as the sum of a selection of the integers $1, 2, 4, 8, \ldots, 2^r, \ldots$ (without repetition) in exactly one way. ∎

Problem 5.4.7. In a partition of any integer n larger than 1 into parts that the powers of $2,$ namely $1, 2, 4, 8, \ldots, 2^r, \ldots,$ prove that the number of partitions that have an even number of parts is equal to the number of partitions that have an odd number of parts.

Solution. Consider

$$\begin{aligned}
1-x &= \frac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2r})}\ldots \\
&= (1-x+x^2-x^3+x^4-\ldots) \\
&\qquad (1-x^2+x^4-x^6+x^8-\ldots) \\
&\qquad (1-x^4+x^8-x^{12}+x^{16}-\ldots)\ldots \\
&\qquad (1-x^{2r}+x^{2.2r}-x^{3.2r}+x^{4.2r}-\ldots)\ldots
\end{aligned}$$

we conclude that to partition any integer n larger than 1 into parts that are powers of $2,$ namely, $1, 2, 4, 8, \ldots, 2^r, \ldots,$ the number of partitions that have an even number of parts is equal to the number of partitions that have an odd number of parts. The series

$$1 - x + x^2 - x^3 + x^4 - \ldots$$

enumerates the number of 1 's in a partition, with terms corresponding to an even number of 1 's in the partition having $+1$ as the coefficients and terms corresponding to an odd number of 1 's in the partition having -1 as the coefficients. Similarly, the series

$$1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

enumerates the number of 2 's in a partition, and the series

$$1 - x^4 + x^8 + x^{12} + x^{16} - \ldots$$

enumerates the number of 4 's in a partition, with terms corresponding to an even number of $2'$ s (or $4'$ s) having positive coefficients and terms corresponding to an odd number of $2'$ s (or $4'$ s) having negative coefficients. Therefore, in the expansion of the product

$$(1 - x + x^2 - x^3 + x^4 - \ldots)(1 - x^2 + x^4 - x^6 + x^8 - \ldots)$$
$$(1 - x^4 + x^8 - x^{12} + x^{16} - \ldots)\ldots$$
$$(1 - x^{2r} + x^{2.2r} - x^{3.2r} + x^{4.2r} - \ldots)\ldots$$

a term $+x^n$ corresponds to a partition of the integer n into an even number of parts, and a term $-x^n$ corresponds to a partition of the integer n into an odd number of parts. ■

Illustration. We see that $4 + 1, 2 + 1 + 1 + 1, 2 + 2 + 1,$ and $1 + 1 + 1 + 1 + 1$ are the four partitions of the integer 5 into parts that are powers of $2.$ Two of these partitions have an even number of parts, and the other two have an odd number of parts.

Exercises

1. Prove the identity $\frac{1}{1-x} = (1 + x + x^2 + \ldots + x^9)(1 + x^{10} + x^{20} + \ldots + x^{90})$ $(1 + x^{100} + x^{200} + \ldots + x^{900})\ldots$
2. Show that the number of partitions of the integer $2r + k$ into exactly $r + k$ parts is the same for any nonnegative integer $k.$
3. Prove that the number of partitions of the integer n into m distinct parts is equal to the number of partitions of the integer $n - [m(m + 1)/2]$ into at most m parts ($n > m(m + 1)/2).$
4. Show that the number of partitions of the integer $2n$ into three parts which are such that the sum of any two parts is greater than the third is equal to the number of

partitions of n into exactly three parts.

5.5 Principle of Inclusion and Exclusion

Let us motivate the subject of this section with a simple illustrative example. In a group of ten girls, six have blond hair, five have blue eyes, and three have blond hair and blue eyes. How many girls are there in the group who have neither blond hair nor blue eyes? Clearly the answer is

$10 - 6 - 5 + 3 = 2$

Since the three blondes with blue eyes are included in the count of the six blondes and are again included in the count of the five with blue eyes, they are subtracted twice in the expression $10 - 6 - 5$. Therefore, 3 should be added to the expression $10 - 6 - 5$ to give the correct count of girls who have neither blond hair nor blue eyes.
The graphical representation in Figure $5.5.1$ shows very clearly the same argument. The area inside the large circle represents the total number of girls. The areas inside the two small circles represent, respectively, the number of girls who have blond hair and the number of girls who have

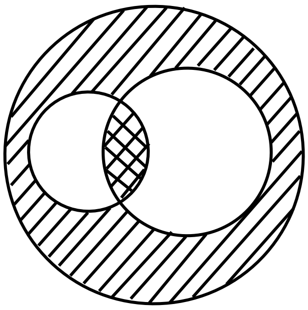


Figure $5.5.1$

blue eyes. The crosshatched area represents the number of girls that have both blond hair and blue eyes. This area is subtracted twice when the areas of the two small circles are subtracted from the area of the large circle. To find the area marked with vertical lines which represents the number of girls who neither are blondes nor have blue eyes, we should, therefore, compensate the oversubtraction by adding back the cross hatched area. The extension of the logical reasoning in this example leads to a very important counting theorem that is studied in this section. To count the number of a certain class of objects, we exclude those that should not be included in the count and, in turn, compensate the count by including those that have been excluded incorrectly. The counting theorem is called the *principle of inclusion and exclusion.*

Principle of Inclusion and Exclusion

Consider a set of N objects. Let $a_1, a_2, \ldots, a_r$ be a set of properties that these objects may have. In general, these properties are not mutually exclusive; that is, an object can have one or more of these properti (The case in which these properties are mutually exclusive proves to be uninteresting special case, as will be seen.) Let $N(a_1)$ denote the number of objects that have the property $a_1,$ let $N(a_2)$ denote the number objects that have the property $a_2, \ldots,$ and let $N(a_r)$ denote the number of objects that have the property $a_r.$ Notice that an object having t property a_i is included in the count $N(a_i)$ regardless of the other propties it may have. Thus, if an object has both the properties a_i and a_j it will contribute a count in $N(a_i)$ as well as a count in $N(a_j).$
Let $N(a_1')$ denote the number of objects that do not have the property $a_1,$ let $N(a_2')$ denote the number of objects that do not have the property $a_2, \ldots,$ and let $N(a_r')$ denote the number of objects that do not have the property $a_r,$ Let $N(a_i a_j)$ denote the number of objects that have both the properties a_i and $a_j,$ let $N(a_i' a_j')$ denote the number of objects that have neither the property a_i nor the property $a_j,$ and let $N(a_i' a_j)$ denote the number of objects that have the property a_j but not the property $a_i.$ Logically, we see that

$$N(a_i') = N - N(a_i)$$

because each of the N objects either has the property a_i [accounted for in $N(a_i)$] or does not have the property a_i [accounted for in $N(a_i')$]. Also,

$$N(a_i' a_j) = N - N(a_i a_j)$$

because for each of the $N(a_j)$ objects that have the property $a_j,$ it either has the property a_i [accounted for in $N(a_i a_j)$] or does not have the property a_i [accounted for in $N(a_i' a_j)$]. Using a similar argument, we have

$$N(a_i' a_j') = N - N(a_i a_j') - N(a_i' a_j) - N(a_i a_j)$$

which can be rewritten as

$$\begin{aligned} N(a_i' a_j') &= N - [N(a_i a_j') + N(a_i a_j)] - [N(a_i' a_j) + [N(a_i a_j)] + N(a_i a_j) \\ &= N - N(a_i) - N(a_j) + N(a_i a_j) \end{aligned}$$

We now prove the following extension of Eqs. (4-1) and (4-2):

$$
\begin{aligned}
N(a_1'a_2'\ldots a_r') = N - N(a_1) - N(a_2) - \ldots - N(a_r) \\
&+ N(a_1a_2) + N(a_1a_3) + \ldots + N(a_{r-1}a_r) \\
&- N(a_1a_2a_3) - N(a_1a_2a_4) - \ldots - N(a_{r-2}a_{r-1}a_r) \\
&+ \ldots \\
&+ (-1)^r N(a_1a_2\ldots a_r) \\
&= N - \sum_i N(a_i) + \sum_{i,j,i\neq j} N(a_ia_j) - \sum_{i,j,k,i\neq j\neq k} N(a_ia_ja_k) \\
&+ \ldots + (-1)^r N(a_1a_2\ldots a_r)
\end{aligned}
$$

This identity, known as the *principle of inclusion and exclusion*, will be proved by induction on the total number of properties the objects may have. As the basis of induction, we have already shown that

$$N(a_1') = N - N(a_1)$$

As the induction hypothesis, we assume that the identity is true for objects having up to $r-1$ properties; that is

$$
\begin{aligned}
N(a_1'a_2'\ldots a_{r-1}') = N - N(a_1) - N(a_2) - \ldots - N(a_{r-1}) \\
&+ N(a_1a_2) + N(a_1a_3) + \ldots + N(a_{r-2}a_{r-1}) \\
&- N(a_1a_2a_3) - N(a_1a_2a_4) - \ldots - N(a_{r-2}a_{r-1}a_r) \\
&+ \ldots \\
&+ (-1)^{r-1} N(a_1a_2\ldots a_{r-1})
\end{aligned}
$$

Now, for a set of N objects having up to r properties, $a_1, a2, \ldots a_r$ we consider the set of $N(a_r)$ objects that have the property a_r. Since this set of objects may have any of the $r-1$ properties $a_1, a_2, \ldots, a_{r-1,}$ according to the induction hypothesis,

$$
\begin{aligned}
N(a_1'a_2'\ldots a_{r-1}'a_r) = N(a_r) - N(a_1a_r) - N(a_2a_r) - \ldots - N(a_{r-1}a_r) \\
&+ N(a_1a_2a_r) + N(a_1a_3a_r) + \ldots + N(a_{r-2}a_{r-1}a_r) \\
&- N(a_1a_2a_3) - N(a_1a_2a_4) - \ldots - N(a_{r-2}a_{r-1}a_r) \\
&+ \ldots
\end{aligned}
$$

$$+ (-1)^{r-1}N(a_1a_2\ldots a_{r-1}a_r)$$

Now,

$$N(a'_1a'_2\ldots a'_{r-1}) - N(a'_1a'_2\ldots a'_{r-1}a_r)$$

$$= N - N(a_1) - N(a_2) - \ldots - N(a_{r-1}) - N(a_r)$$

$$+ N(a_1a_2) + N(a_1a_3) + \ldots + N(a_1a_r)$$

$$+ \ldots$$

$$+ (N(a_{r-1}a_r)$$

$$- \ldots$$

$$+ (-1)^{r-1}N(a_1a_2\ldots a_{r-1}a_r)$$

Thus, $N(a'_1a'_2\ldots a'_{r-1}) - N(a'_1a'_2\ldots a'_{r-1}a_r) = N(a'_1a'_2\ldots a'_{r-1}a'_r)$

Example 5.5.1. Twelve balls are painted in the following way: Two are unpainted. Two are painted red, one is painted blue, and one is painted white. Two are painted red and blue, and one is painted red and white. Three are painted red, blue, and white.
Let $a_1, a_2,$ and a_3 denote the properties that a ball is painted red, blue, and white, respectively; then

$N(a_1) = 8$ $N(a_2) = 6$ $N(a_3) = 5$
$N(a_1a_2) = 5$ $N(a_1a_3) = 4$ $N(a_2a_3) = 3$ It follows that
$N(a_1a_2a_3) = 3$

$$N(a_1a_2a_3) = 12 - 8 - 6 - 5 + 5 + 4 + 3 - 3 = 2.$$

Example 5.5.2. Find the number of integers between 1 and 250 that are not divisible by any of the integers $2, 3, 5,$ and $7.$

Solution Let $a_1, a_2, a_3,$ and a_4 denote the properties that a number is divisible by $2,$ divisibl by $3,$ divisible by $5,$ and divisible by $7,$ respectively. Among the integers 1 through 250 there are 125 integers that are divisible by 2, because every other integer is a multiple of $2.$ Similarly, there are 83 integers that are multiples of 3 and 50 integers that are multiples of 5 and so on.

Letting $[x]$ denote the integral part of the number x,

$N(a_1) = [\frac{250}{2}] = 125$ $\qquad N(a_2) = [\frac{250}{3}] = 83$

$N(a_3) = [\frac{250}{5}] = 50$ $\qquad N(a_4) = [\frac{250}{7}] = 35$

$N(a_1a_2) = [\frac{250}{2\times 3}] = 41$ $\qquad N(a_1a_3) = [\frac{250}{2\times 5}] = 25$

$N(a_1a_4) = [\frac{250}{2\times 7}] = 17$ $\qquad N(a_2a_3) = [\frac{250}{3\times 5}] = 16$

$N(a_2a_4) = [\frac{250}{3\times 7}] = 11$ $\qquad N(a_3a_4) = [\frac{250}{5\times 7}] = 7$

$N(a_1a_2a_3) = [\frac{250}{2\times 3\times 5}] = 8$ $\qquad N(a_1a_2a_4) = [\frac{250}{2\times 3\times 7}] = 5$

$N(a_1a_3a_4) = [\frac{250}{2\times 5\times 7}] = 3$ $\qquad N(a_2a_3a_4) = [\frac{250}{3\times 5\times 7}] = 2$

$N(a_1a_2a_3a_4) = [\frac{250}{2\times 3\times 5\times 7}] = 1$

Therefore, the number of integers that are not divisible by any of the integers $2, 3, 5,$ and 7 is

$$N(a_1'a_2'a_3'a_4') = 250 - (125 + 83 + 50 + 35)$$

$$+ (41 + 25 + 17 + 16 + 11 + 7) - (8 + 5 + 3 + 2) + 1 = 57$$

Similarly, the number of integers that are not divisible by 2 nor by 7 but are divisible by 5 is

$$N(a_1'a_3a_4') = N(a_3) - N(a_1a_3) - N(a_3a_4) + N(a_1a_3a_4)$$

$$= 50 - 25 - 7 + 3$$

$$= 21$$

Problem 5.5.3. Find the number of $r-$ digit quaternary sequences in which each of the three digits $1, 2,$ and 3 appears at least once.

Solution. Let $a_1, a_2,$ and a_3 be the properties that the digits $1, 2,$ and 3 do not appear in a sequence, respectively. Because

$$N(a_1) = N(a_2) = N(a_3) = 3^r$$

$$N(a_1a_2) = N(a_1a_3) = N(a_2a_3) = 2^r$$

$$N(a_1a_2a_3) = 1$$

we have

$$N(a_1'a_2'a_3') = 4^r - 3 \times 3^r + 3 \times 2^r - 1$$

As a matter of fact, using the generating function technique, we derived a formula for the number of ways of distributing r distinct objects into n distinct cells with no cell left empty. This formula can also be derived by the use of the principle of inclusion and exclusion as follows:
Let $a_1, a_2, \ldots, a_n$ be the properties that the 1st,2nd,...,nth cell is left empty in the distributions of the r objects, respectively. Then,

$$N(a_1'a_2' \ldots a_n') = n^r - \binom{n}{1}(n-1)^r + \binom{n}{2}(n-2)^r - \ldots$$

$$+ (-1)^{n-1}\binom{n}{n-1}1^r + (-1)^n\binom{n}{n}0^r$$

$$= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$$

Example 5.5.4. Consider a single ball that is painted with n colours. Let $a_1, a_2, \ldots, a_n$ denote the properties that a ball is painted with the $1st, 2nd, \ldots, nth$ colour, respectively. Since

$$N(a_1) = N(a_2) = \ldots = N(a_n) = 1$$

$$N(a_1a_2) = N(a_1a_3) = \ldots = N(a_{n-1}a_n) = 1$$

$$\ldots$$

$$N(a_1a_2 \ldots a_n) = 1$$

we have

$$N(a_1'a_2' \ldots a_n') = 1 - \binom{n}{1} + \binom{n}{2} - \ldots (-1)^n\binom{n}{n}$$

However, $N(a_1'a_2' \ldots a_n') = 0$
because there is no unpainted ball. Therefore, we have the identity

$$1 - \binom{n}{1} + \binom{n}{2} - \ldots (-1)^n\binom{n}{n} = 0$$

Example 5.5.5. Find the number of permutations of the letters $\alpha,\ \alpha,\ \alpha,\ \beta,\ \beta,\ \beta,\ \gamma,\ \gamma,$ and γ which are such that no identical letters are adjacent.

Solution Let a_1, a_2, a_3 be the properties that α, β, γ are adjacent.

$$N = \frac{9!}{3!3!3!}$$
$$N(a_1) = N(a_2) = N(a_3) = \frac{7!}{3!3!}$$
$$N(a_1a_2) = N(a_2a_3) = N(a_1a_3) = \frac{5!}{3!}$$
$$N(a_1a_2a_3) = 3!$$

By principle of inclusion and exclusion,

$$\begin{aligned} N(a_1'a_2'a_3') &= N - \sum N(a_i) + \sum N(a_ia_j) - N(a_1a_2a_3) \\ &= \frac{9!}{3!3!3!} - 3\frac{7!}{3!3!} + 3\frac{5!}{3!} - 3! \\ &= 1680 - 420 + 60 - 6 \\ &= 1314 \end{aligned}$$

Example 5.5.6. Find the number of permutations of the letters a, b, c, d, e and f in which neither the pattern ace nor the pattern fd appears.

Solution Let a_1 be the property that the pattern ace appears in a permutation, and let a_2 be the property that the pattern fd appears in a permutation.
By the principle of inclusion and exclusion,

$$N(a_1) = 4! \qquad N(a_2) = 5!$$
$$N(a_1a_2) = 3! \qquad N = 6!$$
$$\begin{aligned} N(a_1'a_2') &= N - N(a_1) - N(a_2) + N(a_1a_2) \\ &= 6! - 4! - 5! + 3! \\ &= 582 \end{aligned}$$

Example 5.5.7. In how many ways can the letters $\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta, \gamma$ and γ be arranged so that all the letters of the same kind are not in a single block?

Solution For the permutations of these letters, let a_1 be the property that the four α' s are in one block, let a_2 be the property that the three β' s are in one block, and let a_3 be the property that the two γ' s are in one block.

$$N = \frac{9!}{4!3!2!}$$

$$N(a_1) = \frac{6!}{3!2!} \qquad N(a_1a_2) = \frac{4!}{2!}$$

$$N(a_2) = \frac{7!}{4!2!} \qquad N(a_2a_3) = \frac{6!}{4!}$$

$$N(a_3) = \frac{8!}{4!3!} \qquad N(a_1a_3) = \frac{5!}{3!}$$

$$N(a_1a_2a_3) = 3!$$

By principle of inclusion and exclusion,

$$\begin{aligned} N(a_1'a_2'a_3') &= N - \sum N(a_i) + \sum N(a_ia_j) - N(a_1a_2a_3) \\ &= \frac{9!}{4!3!2!} - \frac{6!}{3!2!} - \frac{7!}{4!2!} - \frac{8!}{4!3!} + \frac{4!}{2!} + \frac{6!}{4!} + \frac{5!}{3!} - 3! \\ &= 871 \end{aligned}$$

Exercises

1. In how many ways can three 0 's, three 1 's, and three 2 's be arranged so that no three adjacent digits are the same in an arrangement?
2. A man has six friends. He has met each of them at dinner 12 times, every two of them six times, every three of them four times, every four of them three times, every five twice, and all six only once. He has dined out eight times without meeting any of them. How many times has he dined out altogether?
3. A symmetric expression in three variables $x, y,$ and z contains nine terms. Four terms contain the variable $x.$ Two terms contain the variables $x, y,$ and $z.$ One term is a constant. How many terms contain the variables x and $y?$
4. Find the number of binary sequences of length 5 in which every 1 is adjacent to another $1.$
5. With three differently colored paints, in how many ways can the walls of a rectangular room be painted so that color changes occur at (and only at) each corner? With two colors?
6. Among the numbers $1, 2, \ldots, 500,$ how many of them are not divisible by 7 but are divisible by 3 or $5?$

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